ISSN 0965-5425, Computational Mathematics and Mathematical Physics, 2018, Vol. 58, No. 5, pp. 761–774. © Pleiades Publishing, Ltd., 2018.
Original Russian Text © N.A. Kazarinov, E.M. Rudoy, V.Yu. Slesarenko, V.V. Shcherbakov, 2018, published in Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, 2018, Vol. 58, No. 5, pp. 790–805.

Mathematical and Numerical Simulation of Equilibrium of an Elastic Body Reinforced by a Thin Elastic Inclusion

N. A. Kazarinov[a,*], **E. M. Rudoy**[a,b,**], **V. Yu. Slesarenko**[a,***], **and V. V. Shcherbakov**[a,b,****]

[a] *Lavrentyev Institute of Hydrodynamics, Siberian Branch, Russian Academy of Sciences, Novosibirsk, 630090 Russia*

[b] *Novosibirsk State University, Novosibirsk, 630090 Russia*

**e-mail: nkazarinov@gmail.com*

***e-mail: rem@hydro.nsc.ru*

****e-mail: slesarenko@gmail.com*

*****e-mail: sherbakov87@gmail.com*

Received March 13, 2017; in final form, March 31, 2017

Abstract—A boundary value problem describing the equilibrium of a two-dimensional linear elastic body with a thin rectilinear elastic inclusion and possible delamination is considered. The stress and strain state of the inclusion is described using the equations of the Euler–Bernoulli beam theory. Delamination means the existence of a crack between the inclusion and the elastic matrix. Nonlinear boundary conditions preventing crack face interpenetration are imposed on the crack faces. As a result, problem with an unknown contact domain is obtained. The problem is solved numerically by applying an iterative algorithm based on the domain decomposition method and an Uzawa-type algorithm for solving variational inequalities. Numerical results illustrating the efficiency of the proposed algorithm are presented.

Keywords: thin elastic inclusion, delamination crack, nonpenetration condition, variational inequality, domain decomposition method, Uzawa algorithm.

DOI: 10.1134/S0965542518050111

1. INTRODUCTION

Artificial materials, such as fiber-reinforced composites, which consist of an elastic matrix reinforced by high-strength thin fibers (inclusions), have been widely used in various engineering applications over the last decades. Experience shows that crack defects, which have a large effect on the strength characteristics of materials, can arise as early as manufacturing or the initial stage of operation of fiber-reinforced composites. This circumstance motivates the development of new mathematical and numerical models describing the deformation and failure of highly inhomogeneous structures with defects of various natures, specifically, fiber-reinforced composites with cracks.

In this paper, we consider a boundary value problem modeling the equilibrium of a two-dimensional linear elastic body with a thin elastic inclusion affected by given surface forces applied to a boundary portion. The thin inclusion is understood as an object whose dimension is lower by one than that of the elastic matrix. It is assumed that the inclusion extends to the outer boundary of the body and is possibly subject to delamination. The underlying model relies on the ideas described in [1] and developed in [2]. The stress and strain state of the inclusion is described using the equations of the Euler–Bernoulli beam theory. Delamination means the existence of a crack between the inclusion and the elastic matrix. On the crack faces, we set nonlinear boundary conditions in the form of equalities and inequalities preventing the interpenetration of the crack faces. As a result, a problem with an unknown contact area arises. It is shown that the equilibrium problem under consideration is equivalent to a variational inequality on the set of kinematically admissible constraints. For the numerical solution of the problem, an iterative algorithm is proposed based on the domain decomposition method and an Uzawa-type algorithm for solving variational inequalities. Every iteration step consists of solving four linear problems, namely, two plane elasticity problems, the problem of stretching an elastic rod, and a beam bending problem. The solutions of these four problems are related by Lagrange multipliers. The algorithm designed is tested using the finite element method.

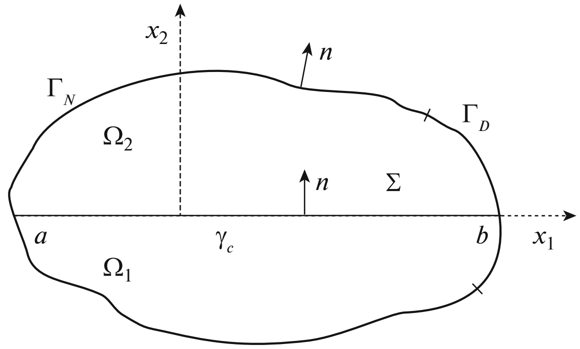

Fig. 1. Geometry of the problem.

Let us briefly overview the works that are closest in subject to our study. In describing the deformation and failure of fiber-reinforced composites, it is reasonable to use not only models of elastic inclusions, but also models of semi-rigid and rigid inclusions. In this case, some (possibly all) of the equilibrium equations for the elastic inclusions are replaced by nonlocal boundary conditions ensuring equilibrium conditions for the inclusions. In recent publications [3–5], well-posed mathematical models for the equilibrium of elastic bodies with thin elastic, rigid, and semi-rigid inclusions and cracks with boundary conditions ensuring mutual nonpenetration of the crack faces were described and the dependence of the solutions on the stiffness parameters of the inclusions was analyzed. Results of numerical simulation concerning the equilibrium of elastic bodies with thin rigid inclusions can be found in [6–10]. The domain decomposition method as applied to various problems in continuum and solid mechanics was discussed in [11–16] and [17–19], respectively. Note also [8, 20, 21], where problems in the theory of cracks with possibly contacting crack faces were solved numerically by applying the domain decomposition method. Other methods for solving such problems were developed in [22–24].

2. FORMULATION OF THE PROBLEM

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$, where $\partial\Omega = \overline{\Gamma}_N \cup \overline{\Gamma}_D$, $\Gamma_N \cap \Gamma_D = \emptyset$, $\operatorname{meas}\Gamma_D > 0$, and $\Sigma = (a,b)\times\{0\}$ is a segment in Ω with its endpoints $(a,0)$ and $(b,0)$ belonging to the outer boundary $\partial\Omega$ so that $(a,0)$ lies on Γ_N, and $(b,0)$, on Γ_D, as shown in Fig. 1. The segment Σ divides Ω into two subdomains Ω_1 and Ω_2 with Lipschitz boundaries $\partial\Omega_\alpha$ such that $\operatorname{meas}(\partial\Omega_\alpha \cap \Gamma_D) > 0$, $\alpha = 1,2$. Suppose that $n = (n_1, n_2)$ denotes the unit outward normal to $\partial\Omega$, $\nu = (0,1)$ denotes the unit normal to Σ, and $\tau = (1,0)$. Finally, let $\Omega_\Sigma = \Omega\backslash\overline{\Sigma}$.

In our consideration, the domain Ω_Σ corresponds to an elastic body made of a generally inhomogeneous anisotropic material with an elasticity tensor $C = \{c_{ijkl}\}$, $i,j,k,l = 1,2$, having the standard properties of symmetry and positive definiteness and satisfying $c_{ijkl} \in L^\infty(\Omega)$; and Σ represents a thin elastic inclusion whose behavior is described using the model of the Euler–Bernoulli elastic beam [25] and the mechanical and geometric properties are characterized by Young's modulus E of the material, by the cross section moment of inertia I, and by the area S of the inclusion. Assume that Σ delaminates from the elastic part Ω_2 on the segment γ_c. Thus, there is a crack between the elastic body and the inclusion. The following variants are also possible: $\gamma_c = \emptyset$ (no delamination) or $\gamma_c = \Sigma$ (complete delamination). In the model under consideration, the conditions on the crack faces are set in the form of equalities and inequalities preventing mutual penetration of the crack faces (see, e.g., [26]).

The elastic behavior of the material is described using the theory of small deformations, i.e., Hooke's law is used as an equation of state and the strain tensor $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$ is connected with the displacement vector $u = (u_1, u_2)$ by the linear relations

$$\sigma(u) = C\varepsilon(u), \quad \varepsilon_{ij}(u) = 1/2(u_{i,j} + u_{j,i}), \quad i,j = 1,2.$$

The subscript after the comma denotes differentiation with respect to the corresponding coordinate.

Consider the following mixed boundary value problem: given a vector of surface forces $f = (f_1, f_2) \in L_2(\Gamma_N)^2$, find the displacement field u for the points of the elastic body, horizontal displacements v, and vertical deflections w of the points of the thin elastic inclusion such that

$$-\operatorname{div}\sigma(u) = 0 \quad \text{in} \quad \Omega_\Sigma, \tag{1}$$

$$-ESv_{,11} = [\sigma_\tau(u)] \quad \text{on} \quad \Sigma, \tag{2}$$

$$EIw_{,1111} = [\sigma_\nu(u)] \quad \text{on} \quad \Sigma, \tag{3}$$

$$u = 0 \quad \text{on} \quad \Gamma_D, \tag{4}$$

$$\sigma(u)n = f \quad \text{on} \quad \Gamma_N, \tag{5}$$

$$v_{,1} = w_{,11} = w_{,111} = 0 \quad \text{at} \quad x_1 = a, \tag{6}$$

$$v = w = w_{,1} = 0 \quad \text{at} \quad x_1 = b, \tag{7}$$

$$[u]\cdot\nu \geq 0, \quad \sigma_\nu(u)\big|_{\gamma_c^2}([u]\cdot\nu) \quad \text{on} \quad \gamma_c, \tag{8}$$

$$\sigma_\nu(u)\big|_{\gamma_c^2} \leq 0, \quad \sigma_\tau(u)\big|_{\gamma_c^2} = 0 \quad \text{on} \quad \gamma_c, \tag{9}$$

$$v = u\cdot\tau, \quad w = u\cdot\nu \quad \text{on} \quad \Sigma^1, \tag{10}$$

$$[u] = 0 \quad \text{on} \quad \Sigma\setminus\overline{\gamma}_c. \tag{11}$$

Here, $[h] = h\big|_{\Sigma^2} - h\big|_{\Sigma^1}$ is the jump in h across Σ, where $h\big|_{\Sigma^\alpha}$ is the trace of h on Σ taken on the side of Ω_α, $\alpha = 1, 2$; $\sigma_\nu = (\sigma(u)\nu)\cdot\nu$; and $\sigma_\tau = (\sigma(u)\nu)\cdot\tau$. Functions defined on Σ are identified with functions of variable x_1.

Relations (1) are equilibrium equations for the elastic body, while (2) and (3) are the Euler–Bernoulli equations for the thin elastic inclusion Σ. The right-hand sides $[\sigma_\tau(u)]$ and $[\sigma_\nu(u)]$ of equilibrium equations (2) and (3) describe the forces exerted on Σ by the ambient elastic medium. The inequality in (8) ensures the mutual nonpenetration of the crack faces. If there is no contact at a given point $x_* \in \gamma_c$ ($[u]\cdot\nu(x_*) > 0$), then the normal stress on the crack face γ_c^2 is zero ($\sigma_\nu(u)\big|_{\gamma_c^2}(x_*) = 0$). On the other hand, if the normal stress on γ_c^2 is negative ($\sigma_\nu(u)\big|_{\gamma_c^2}(x_*) < 0$), then the crack faces are in contact ($[u]\cdot\nu(x_*) = 0$). Moreover, conditions (10) guarantee that the displacements of the elastic body on Σ^1 are equal to those of the thin inclusion. The boundary conditions (6) mean that the corresponding end of the thin inclusion is free, while conditions (7) correspond to a clamped end. Note once again that the contact points of the opposite crack faces are not known a priori and are to be determined in the course of problem solving.

3. VARIATIONAL FORMULATION OF THE PROBLEM

To obtain a variational formulation of the boundary value problem (1)–(11), we introduce the function spaces

$$H^1_{\Gamma_D}(\Omega_c)^2 = \left\{u \in H^1(\Omega_c)^2 \,|\, u = 0 \text{ on } \Gamma_D\right\},$$

$$H^1_b(\Sigma) = \left\{v \in H^1(\Sigma) \,|\, v = 0 \text{ at } x_1 = b\right\},$$

$$H^2_b(\Sigma) = \left\{w \in H^2(\Sigma) \,|\, w = w_{,1} = 0 \text{ at } x_1 = b\right\}$$

and the set of kinematically admissible displacements

$$K = \{(u, v, w) \in H^1_{\Gamma_D}(\Omega_c)^2 \times H^1_b(\Sigma) \times H^2_b(\Sigma) \mid [u]\cdot\nu \geq 0 \quad \text{on} \quad \gamma_c,$$
$$v = u\cdot\tau, \quad w = u\cdot\nu \quad \text{on} \quad \Sigma^1, \quad [u] = 0 \quad \text{on} \quad \Sigma\setminus\overline{\gamma}_c\},$$

where $\Omega_c = \Omega \setminus \overline{\gamma}_c$. The potential energy functional of the system is defined by the formula

$$\Pi(u, v, w) = \frac{1}{2}\int_{\Omega_c} \sigma(u) : \varepsilon(u) dx + \frac{1}{2}\int_{\Sigma} ESv_{,1}^2 dx_1 + \frac{1}{2}\int_{\Sigma} EIw_{,11}^2 dx_1 - \int_{\Gamma_N} f \cdot u ds.$$

Consider the problem of minimizing Π over the set K: find an element $(u, v, w) \in K$ such that

$$\Pi(u, v, w) \le \Pi(\overline{u}, \overline{v}, \overline{w}) \quad \forall (\overline{u}, \overline{v}, \overline{w}) \in K. \tag{12}$$

Theorem 1. *There exists a unique solution* $(u, v, w) \in K$ *of problem* (12). *This solution satisfies the variational inequality*

$$\int_{\Omega_c} \sigma(u) : \varepsilon(\overline{u} - u) dx + \int_{\Sigma} ESv_{,1}(\overline{v}_{,1} - v_{,1}) dx_1 + \int_{\Sigma} EIw_{,11}(\overline{w}_{,11} - w_{,11}) dx_1 \\ \ge \int_{\Gamma_N} f \cdot (\overline{u} - u) ds \quad \forall (\overline{u}, \overline{v}, \overline{w}) \in K. \tag{13}$$

Moreover, the variational inequality (13) *represents a weak formulation of the boundary value problem* (1)–(11).

Proof. The set K is convex and closed (and, hence, weakly closed) in the reflexive space $H^1_{\Gamma_D}(\Omega_c)^2 \times H^1_b(\Sigma) \times H^2_b(\Sigma)$, while the functional Π is weakly lower semicontinuous and coercive over this space. The last assertion is a simple consequence of the first Korn inequality and the Poincaré–Friedrichs inequality. Therefore, problem (12) has a solution (see [26, Theorem 1.11]). The uniqueness is checked directly. Note also that the variational inequality (13) gives a necessary and sufficient condition for the convex and Gâteaux differentiable functional Π to have a minimum over the set K.

Let us verify that the boundary value problem (1)–(11) is equivalent to variational inequality (13) for smooth solutions. First, assume that relations (1)–(11) hold. Choosing $(\overline{u}, \overline{v}, \overline{w}) \in K$, multiplying the equilibrium equations (1), (2) and (3) by $(\overline{u} - u)$, $(\overline{v} - v)$, and $(\overline{w} - w)$, respectively, and integrating the results over Ω_Σ and Σ, we obtain

$$-\int_{\Omega_\Sigma} \operatorname{div} \sigma(u) \cdot (\overline{u} - u) dx + \int_{\Sigma} (ESv_{,11} - [\sigma_\tau(u)])(\overline{v} - v) dx_1 + \int_{\Sigma} (EIw_{,1111} - [\sigma_\nu(u)])(\overline{w} - w) dx_1 = 0.$$

Integration by parts, in view of (4)–(7) and (11), yields

$$\int_{\Omega_\Sigma} \sigma(u) : \varepsilon(\overline{u} - u) dx + \int_{\gamma_c} [\sigma(u)\nu \cdot (\overline{u} - u)] dx_1 + \int_{\Sigma \setminus \overline{\gamma}_c} [\sigma(u)\nu] \cdot (\overline{u} - u) dx_1 + \int_{\Sigma} ESv_{,1}(\overline{v}_{,1} - v_{,1}) dx_1 \\ + \int_{\Sigma} EIw_{,11}(\overline{w}_{,11} - w_{,11}) dx_1 - \int_{\Sigma} [\sigma_\tau(u)](\overline{v} - v) dx_1 - \int_{\Sigma} [\sigma_\nu(u)](\overline{w} - w) dx_1 - \int_{\Gamma_N} f \cdot (\overline{u} - u) ds = 0.$$

Once again using (11), we note that the integration over Ω_Σ can be replaced by integration over Ω_c. Thus, to obtain variational inequality (13), it is sufficient to prove

$$\int_{\gamma_c} [\sigma(u)\nu \cdot (\overline{u} - u)] dx_1 - \int_{\gamma_c} [\sigma_\tau(u)](\overline{v} - v) dx_1 - \int_{\gamma_c} [\sigma_\nu(u)](\overline{w} - w) dx_1 \le 0.$$

Obviously, the last inequality holds by virtue of boundary conditions (8)–(10).

Now, we prove the converse. Assume that variational inequality (13) is satisfied. It is easy to see that the equilibrium equations (1) hold in the sense of distributions. To check this assertion, it is sufficient to substitute $(\overline{u}, \overline{v}, \overline{w}) = (u, v, w) \pm (\breve{u}, 0, 0)$, $\breve{u} \in C_0^\infty(\Omega_\Sigma)^2$, as test functions into (13). Then, using $(\overline{u}, \overline{v}, \overline{w}) = (u, v, w) \pm (\hat{u}, \hat{v}, \hat{w})$, $[\hat{u}] \cdot \tau = 0$ on $\Sigma \setminus \overline{\gamma}_c$ and $[\hat{u}] \cdot \nu = 0$, $\hat{u}|_{\Sigma^1} \cdot \tau = \hat{v}$, $\hat{u}|_{\Sigma^1} \cdot \nu = \hat{w}$ on Σ as test functions in (13), we have

$$\int_{\Omega_c} \sigma(u) : \varepsilon(\hat{u}) dx + \int_{\Sigma} ESv_{,1}\hat{v}_{,1} dx_1 + \int_{\Sigma} EIw_{,11}\hat{w}_{,11} dx_1 - \int_{\Gamma_N} f \cdot \hat{u} ds = 0. \tag{14}$$

Integration by parts in (14) yields

$$-\int_{\gamma_c}[\sigma(u)\nu \cdot \hat{u}]dx_1 - \int_{\Sigma\setminus\overline{\gamma}_c}[\sigma(u)\nu]\cdot \hat{u}dx_1 - \int_{\Sigma} ESv_{,11}\hat{v}dx_1 + \int_{\Sigma} EIw_{,1111}\hat{w}dx_1$$
$$- ESv_{,1}\hat{v}(a) - EIw_{,11}\hat{w}_{,1}(a) + EIw_{,111}\hat{w}(a) - \int_{\Gamma_N} f\cdot\hat{u}ds = 0. \tag{15}$$

Assuming that $\hat{v} = \hat{w} = \hat{w}_{,1} = 0$ for $x_1 = b$ and $\hat{u} = 0$ on Γ_N, we derive

$$-\int_{\gamma_c}\left([\sigma_\nu(u)]\hat{u}\cdot\nu + [\sigma_\tau(u)\hat{u}\cdot\tau]\right)dx_1 - \int_{\Sigma\setminus\overline{\gamma}_c}\left([\sigma_\nu(u)]\hat{u}\cdot\nu + [\sigma_\tau(u)]\hat{u}\cdot\tau\right)dx_1$$
$$- \int_{\Sigma} ESv_{,11}\hat{v}dx_1 + \int_{\Sigma} EIw_{,1111}\hat{w}dx_1 = 0. \tag{16}$$

Since $\hat{u}\cdot\tau$ on γ_c^2 is arbitrary, we conclude that $\sigma_\tau(u)|_{\gamma_c^2} = 0$. Since $v = u\cdot\tau$ and $w = u\cdot\nu$ on Σ^1, the equilibrium equations (2) and (3) follow from (16). Taking into account these relations and again using (15), we see that the boundary conditions (5) and (6) are satisfied. It remains to check the boundary conditions from (8) and (9) for the normal stress $\sigma_\nu(u)|_{\gamma_c^2}$. The corresponding argument is omitted, since it repeats, word for word, the proof in [2] for the case of both free ends of Σ lying inside Ω. The theorem is proved.

4. DECOMPOSITION OF THE PROBLEM

Consider the function spaces

$$U^\alpha = \{u^\alpha \in H^1(\Omega_\alpha)^2 \,|\, u^\alpha = 0 \text{ on } \partial\Omega_\alpha \cap \Gamma_D\}, \quad \alpha = 1,2,$$
$$V = H_b^1(\Sigma), \quad W = H_b^2(\Sigma),$$

and the convex closed (in $U^1\times U^2\times V\times W$) set

$$K_{gc} = \{(u^1,u^2,v,w)\in U^1\times U^2\times V\times W \,|\, (u^2-u^1)\cdot\nu \ge 0 \text{ on } \gamma_c,\ v - u^1\cdot\tau = 0,$$
$$w - u^1\cdot\nu = 0 \text{ on } \Sigma, \quad (u^2-u^1)\cdot\tau = 0, \quad (u^2-u^1)\cdot\nu = 0 \text{ on } \Sigma\setminus\overline{\gamma}_c\}.$$

The energy functionals are defined as

$$\Pi_\alpha(u^\alpha) = \frac{1}{2}\int_{\Omega_\alpha}\sigma(u^\alpha):\varepsilon(u^\alpha)dx - \int_{\partial\Omega_\alpha\cap\Gamma_N} f\cdot u^\alpha ds, \quad \alpha = 1,2,$$
$$\pi_\tau(v) = \frac{1}{2}\int_{\Sigma} ESv_{,1}^2dx_1, \quad \pi_\nu(w) = \frac{1}{2}\int_{\Sigma} EIw_{,11}^2dx_1.$$

Consider the following minimization problem: find an element $(u^1,u^2,v,w)\in K_{gc}$ such that

$$\Pi_1(u^1) + \Pi_2(u^2) + \pi_\tau(v) + \pi_\nu(w) \le \Pi_1(\overline{u}^1) + \Pi_2(\overline{u}^2) + \pi_\tau(\overline{v}) + \pi_\nu(\overline{w}) \quad \forall(\overline{u}^1,\overline{u}^2,\overline{v},\overline{w})\in K_{gc}. \tag{17}$$

As before, it can be shown that problem (17) has a unique solution $(u^1,u^2,v,w)\in K_{gc}$ satisfying the variational inequality

$$\int_{\Omega_1}\sigma(u^1):\varepsilon(\overline{u}^1-u^1)dx + \int_{\Omega_2}\sigma(u^2):\varepsilon(\overline{u}^2-u^2)dx + \int_{\Sigma} ESv_{,1}(\overline{v}_{,1}-v_{,1})dx_1 + \int_{\Sigma} EIw_{,11}(\overline{w}_{,11}-w_{,11})dx_1$$
$$\ge \int_{\partial\Omega_1\cap\Gamma_N} f\cdot(\overline{u}^1-u^1)ds + \int_{\partial\Omega_2\cap\Gamma_N} f\cdot(\overline{u}^2-u^2)ds \quad \forall(\overline{u}^1,\overline{u}^2,\overline{v},\overline{w})\in K_{gc}. \tag{18}$$

Moreover, we have

$$u^\alpha = u|_{\Omega_\alpha}, \quad \alpha = 1,2,$$

where (u,v,w) is a solution of the minimization problem (12).

5. APPROXIMATE PROBLEM

To solve the problem with inequality constraints (17), we properly regularize it. For this purpose, we apply the approach used in [27, Chapter 5] for solving the problem of elastoplastic torsion of a rod and adapted in [8, 20, 28] to solving crack problems with possibly contacting crack faces. Specifically, for an arbitrary number $p > 0$, define the sets

$$\Lambda_p^1 = \{\lambda \in L_2(\gamma_c) | 0 \le \lambda \le p \text{ on } \gamma_c\},$$

$$\Lambda_p^2 = \Lambda_p^3 = \{\lambda \in L_2(\Sigma) | -p \le \lambda \le p \text{ on } \Sigma\},$$

$$\Lambda_p^4 = \Lambda_p^5 = \{\lambda \in L_2(\Sigma\backslash\overline{\gamma}_c) | -p \le \lambda \le p \text{ on } \Sigma\backslash\overline{\gamma}_c\}.$$

Over the set $U^1 \times U^2 \times V \times W \times \Lambda_p^1 \times \Lambda_p^2 \times \Lambda_p^3 \times \Lambda_p^4 \times \Lambda_p^5$, consider the Lagrange functional

$$\mathcal{L}(u^1, u^2, v, w, \lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5) = \Pi_1(u^1) + \Pi_2(u^2) + \pi_\tau(v) + \pi_\nu(w) + \int_{\gamma_c} \lambda^1(u^1 - u^2)\cdot \nu dx_1$$
$$+ \int_\Sigma \lambda^2(u^1\cdot\tau - v)dx_1 + \int_\Sigma \lambda^3(u^1\cdot\nu - w)dx_1 + \int_{\Sigma\backslash\overline{\gamma}_c} \lambda^4(u^1 - u^2)\cdot\nu dx_1 + \int_{\Sigma\backslash\overline{\gamma}_c} \lambda^5(u^1 - u^2)\cdot\tau dx_1,$$

which is associated with the following family of saddle point problems depending on the regularization parameter p: find an element $(u^1, u^2, v, w, \lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5) \in U^1 \times U^2 \times V \times W \times \Lambda_p^1 \times \Lambda_p^2 \times \Lambda_p^3 \times \Lambda_p^4 \times \Lambda_p^5$ such that

$$\begin{gathered}\mathcal{L}(u^1, u^2, v, w, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\lambda}^3, \overline{\lambda}^4, \overline{\lambda}^5) \le \mathcal{L}(u^1, u^2, v, w, \lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5) \le \mathcal{L}(\overline{u}^1, \overline{u}^2, \overline{v}, \overline{w}, \lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5)\\ \forall(\overline{u}^1, \overline{u}^2, \overline{v}, \overline{w}, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\lambda}^3, \overline{\lambda}^4, \overline{\lambda}^5) \in U^1 \times U^2 \times V \times W \times \Lambda_p^1 \times \Lambda_p^2 \times \Lambda_p^3 \times \Lambda_p^4 \times \Lambda_p^5.\end{gathered} \tag{19}$$

The Lagrange multipliers λ^i, $i = \overline{1,5}$, thus introduced can be interpreted as follows: λ^1 ensures the validity of the nonpenetration condition for the crack faces γ_c, λ^2 and λ^3 relate the displacement fields of Σ and Ω_1, and λ^4 and λ^5 relate the displacement fields of Ω_1 and Ω_2 outside γ_c.

Since each the sets in the direct product $U^1 \times U^2 \times V \times W \times \Lambda_p^1 \times \Lambda_p^2 \times \Lambda_p^3 \times \Lambda_p^4 \times \Lambda_p^5$ is convex, closed, and bounded in the corresponding reflexive space, while the Lagrange functional L is convex and lower semicontinuous on $U^1 \times U^2 \times V \times W$ and concave and upper semicontinuous on $\Lambda_p^1 \times \Lambda_p^2 \times \Lambda_p^3 \times \Lambda_p^4 \times \Lambda_p^5$, the theory of existence of saddle points guarantees that, for every $p > 0$, problem (19) has at least one solution (see [29, Chapter 6, Proposition 2.1]).

Theorem 2. *Let* (u^1, u^2, v, w) *be a solution of problem* (17) *and* $(u_p^1, u_p^2, v_p, w_p, \lambda_p^1, \lambda_p^2, \lambda_p^3, \lambda_p^4, \lambda_p^5)$ *be a solution of problem* (19). *Then, as* $p \to \infty$,

$$(u_p^1, u_p^2, v_p, w_p) \to (u^1, u^2, v, w) \quad \text{strongly in} \quad U^1 \times U^2 \times V \times W.$$

Proof. Inequalities (19) are equivalent to the following system of variational equalities and inequalities:

$$\int_{\Omega_1} \sigma(u_p^1) : \varepsilon(\overline{u}^1)dx + \int_{\Omega_2} \sigma(u_p^2) : \varepsilon(\overline{u}^2)dx + \int_\Sigma ESv_{p,1}\overline{v}_{,1}dx_1 + \int_\Sigma EIw_{p,11}\overline{w}_{,11}dx_1 + \int_{\gamma_c} \lambda_p^1(\overline{u}^1 - \overline{u}^2\overline{u}^2)\cdot\nu dx_1$$
$$+ \int_\Sigma \lambda_p^2(\overline{u}^1\cdot\tau - \overline{v})dx_1 + \int_\Sigma \lambda_p^3(\overline{u}^1\cdot\nu - \overline{w})dx_1 + \int_{\Sigma\backslash\overline{\gamma}_c} da_p^4(\overline{u}^1 - \overline{u}^2)\cdot\nu dx_1 + \int_{\Sigma\backslash\overline{\gamma}_c} \lambda_p^5(\overline{u}^1 - \overline{u}^2)\cdot\tau dx_1 \tag{20}$$
$$= \int_{\partial\Omega_1\cap\Gamma_N} f\cdot\overline{u}^1 ds + \int_{\partial\Omega_2\cap\Gamma_N} f\cdot\overline{u}^2 ds \quad \forall(\overline{u}^1, \overline{u}^2, \overline{v}, \overline{w}) \in U^1 \times U^2 \times V \times W,$$

$$\int_{\gamma_c} \overline{\lambda}^1(u_p^1 - u_p^2)\cdot\nu dx_1 \le \int_{\gamma_c} \lambda_p^1(u_p^1 - u_p^2)\cdot\nu dx_1 \quad \forall\overline{\lambda}^1 \in \Lambda_p^1, \tag{21}$$

$$\int_\Sigma \overline{\lambda}^2(u_p^1\cdot\tau - v_p)dx_1 \le \int_\Sigma \lambda_p^2(u_p^1\cdot\tau - v_p)dx_1 \quad \forall\overline{\lambda}^2 \in \Lambda_p^2, \tag{22}$$

$$\int_\Sigma \overline{\lambda}^3(u_p^1\cdot\nu - w_p)dx_1 \le \int_\Sigma \lambda_p^3(u_p^1\cdot\nu - w_p)dx_1 \quad \forall\overline{\lambda}^3 \in \Lambda_p^3, \tag{23}$$

$$\int\limits_{\Sigma\setminus\bar{\gamma}_c} \bar{\lambda}^4(u_p^1 - u_p^2)\cdot\nu dx_1 \le \int\limits_{\Sigma\setminus\bar{\gamma}_c} \lambda_p^4(u_p^1 - u_p^2)\cdot\nu dx_1 \quad \forall\bar{\lambda}^4 \in \Lambda_p^4, \tag{24}$$

$$\int\limits_{\Sigma\setminus\bar{\gamma}_c} \bar{\lambda}^5(u_p^1 - u_p^2)\cdot\tau dx_1 \le \int\limits_{\Sigma\setminus\bar{\gamma}_c} \lambda_p^5(u_p^1 - u_p^2)\cdot\tau dx_1 \quad \forall\bar{\lambda}^5 \in \Lambda_p^5. \tag{25}$$

It follows from (21)–(25) that the components of the vector function $(\lambda_p^1, \lambda_p^2, \lambda_p^3, \lambda_p^4, \lambda_p^5)$ can be represented in the form

$$\lambda_p^1(x) = \begin{cases} 0, & (u_p^1(x) - u_p^2(x))\cdot\nu \le 0, \\ p, & (u_p^1(x) - u_p^2(x))\cdot\nu > 0, \end{cases} \tag{26}$$

$$\lambda_p^2(x) = \begin{cases} -p, & (u_p^1(x)\cdot\tau - v(x_1)) \le 0, \\ p, & (u_p^1(x)\cdot\tau - v(x_1)) > 0, \end{cases} \tag{27}$$

$$\lambda_p^3(x) = \begin{cases} -p, & (u_p^1(x)\cdot\nu - w(x_1)) \le 0, \\ p, & (u_p^1(x)\cdot\nu - w(x_1)) > 0, \end{cases} \tag{28}$$

$$\lambda_p^4(x) = \begin{cases} -p, & (u_p^1(x) - u_p^2(x))\cdot\nu \le 0, \\ p, & (u_p^1(x) - u_p^2(x))\cdot\nu > 0, \end{cases} \tag{29}$$

$$\lambda_p^5(x) = \begin{cases} -p, & (u_p^1(x) - u_p^2(x))\cdot\tau \le 0, \\ p, & (u_p^1(x) - u_p^2(x))\cdot\tau > 0. \end{cases} \tag{30}$$

Let

$$I_p^1 = \int\limits_{\gamma_c} \max\{0, (u_p^1 - u_p^2)\cdot\nu\} dx_1,$$

$$I_p^2 = \int\limits_{\Sigma} \max\{0, u_p^1\cdot\tau - v_p\} dx_1 + \int\limits_{\Sigma} \max\{0, -(u_p^1\cdot\tau - v_p)\} dx_1,$$

$$I_p^3 = \int\limits_{\Sigma} \max\{0, u_p^1\cdot\nu - w_p\} dx_1 + \int\limits_{\Sigma} \max\{0, -(u_p^1\cdot\nu - w_p)\} dx_1,$$

$$I_p^4 = \int\limits_{\Sigma\setminus\bar{\gamma}_c} \max\{0, (u_p^1 - u_p^2)\cdot\nu\} dx_1 + \int\limits_{\Sigma} \max\{0, -(u_p^1 - u_p^2)\cdot\nu\} dx_1,$$

$$I_p^5 = \int\limits_{\Sigma\setminus\bar{\gamma}_c} \max\{0, (u_p^1 - u_p^2)\cdot\tau\} dx_1 + \int\limits_{\Sigma} \max\{0, -(u_p^1 - u_p^2)\cdot\tau\} dx_1.$$

Then, by virtue of (26)–(30), we have

$$\int\limits_{\gamma_c} \lambda_p^1(u_p^1 - u_p^2)\cdot\nu dx_1 = pI_p^1 \ge 0, \quad \int\limits_{\Sigma} \lambda_p^2(u_p^1\cdot\tau - v_p) dx_1 = pI_p^2 \ge 0, \tag{31}$$

$$\int\limits_{\Sigma} \lambda_p^3(u_p^1\cdot\nu - w_p) dx_1 = pI_p^3 \ge 0, \quad \int\limits_{\Sigma\setminus\bar{\gamma}_c} \lambda_p^4(u_p^1 - u_p^2)\cdot\nu dx_1 = pI_p^4 \ge 0, \tag{32}$$

$$\int\limits_{\Sigma\setminus\bar{\gamma}_c} \lambda_p^5(u_p^1 - u_p^2)\cdot\tau dx_1 = pI_p^5 \ge 0. \tag{33}$$

Note that, by virtue of the first Korn inequality and the Poincaré–Friedrichs inequality, the norms in the spaces U^α $(\alpha = 1, 2)$, V, and W can be defined as

$$\|u^\alpha\|_{U^\alpha}^2 = \int\limits_{\Omega_\alpha} \sigma(u^\alpha) : \varepsilon(u^\alpha) dx, \quad \|v\|_V^2 = 2\pi_\tau(v), \quad \|w\|_V^2 = 2\pi_\nu(w).$$

The variational equality (20) implies that

$$\left\|u_p^1\right\|_{U^1}^2 + \left\|u_p^2\right\|_{U^2}^2 + \left\|v_p\right\|_V^2 + \left\|w_p\right\|_W^2 + pI_p^1 + pI_p^2 + pI_p^3 + pI_p^4 + pI_p^5 = \int\limits_{\partial\Omega_1\cap\Gamma_N} f \cdot u_p^1 ds + \int\limits_{\partial\Omega_2\cap\Gamma_N} f \cdot u_p^2 ds. \quad (34)$$

Taking into account estimates (31)–(33) and the Cauchy inequality, we obtain the p-uniform estimate

$$\left\|u_p^1\right\|_{U^1} + \left\|u_p^2\right\|_{U^2} + \left\|v_p\right\|_V + \left\|w_p\right\|_W \le C. \quad (35)$$

Moreover, for all $p > 0$, it follows from (34) and (35) that

$$0 \le I_p^1 + I_p^2 + I_p^3 + I_p^4 + I_p^5 \le \frac{C}{p}, \quad (36)$$

whence

$$\lim_{p\to\infty}(I_p^1 + I_p^2 + I_p^3 + I_p^4 + I_p^5) = 0. \quad (37)$$

On the basis of (35), we choose a subsequence of the sequence $\left\{(u_p^1, u_p^2, v_p, w_p)\right\}$ (retaining the same notation) such that, as $p \to \infty$,

$$(u_p^1, u_p^2, v_p, w_p) \to (\tilde{u}^1, \tilde{u}^2, \tilde{v}, \tilde{w}) \quad \text{weakly in} \quad U^1 \times U^2 \times V \times W. \quad (38)$$

Let us check that the limiting function $(\tilde{u}^1, \tilde{u}^2, \tilde{v}, \tilde{w})$ thus defined belongs to K_{gc}. Indeed, by virtue of (37), an arbitrary nonnegative function φ of the class $C_0^\infty(\gamma_c)$ satisfies the relations

$$\int\limits_{\gamma_c} \varphi(\tilde{u}^1 - \tilde{u}^2) \cdot \nu dx_1 = \lim_{p\to\infty} \int\limits_{\gamma_c} \varphi(u_p^1 - u_p^2) \cdot \nu dx_1 \le \left(\max_{x\in\overline{\gamma}_c} \eta(x)\right) \lim_{p\to\infty} \int\limits_{\gamma_c} \max\{0, (u_p^1 - u_p^2) \cdot \nu\} dx_1$$
$$= \left(\max_{x\in\overline{\gamma}_c} \varphi(x)\right) \lim_{p\to\infty} I_p^1 = 0;$$

therefore, $(\tilde{u}^2 - \tilde{u}^1) \cdot \nu \ge 0$ on γ_c. Furthermore, for an arbitrary function ψ of the class $C_0^\infty(\Sigma)$, we derive

$$\int\limits_\Sigma (\tilde{u}^1 \cdot \tau - \tilde{v})\psi dx_1 = \lim_{p\to\infty} \int\limits_\Sigma (u_p^1 \cdot \tau - v_p)\psi dx_1.$$

Moreover, the following chain of inequalities holds:

$$-\left(\max_{x\in\overline{\Sigma}} |\psi(x)|\right) \int\limits_\Sigma \max\{0, -(u_p^1 \cdot \tau - v_p)\} dx_1 \le \int\limits_\Sigma (u_p^1 \cdot \tau - v_p)\psi dx_1 \le \left(\max_{x\in\overline{\Sigma}} |\psi(x)|\right) \int\limits_\Sigma \max\{0, u_p^1 \cdot \tau - v_p\} dx_1. \quad (39)$$

Taking into account estimate (37) and passing to the limit as $p \to \infty$ in (39), we obtain

$$\int\limits_\Sigma (\tilde{u}^1 \cdot \tau - \tilde{v})\psi dx_1 = 0.$$

This equality means that $\tilde{u}^1 \cdot \tau - \tilde{v} = 0$ on Σ. A similar argument yields $\tilde{u}^1 \cdot \nu - \tilde{w} = 0$ on Σ and $(u^2 - u^1) \cdot \nu = 0$, $(u^2 - u^1) \cdot \tau = 0$ on $\Sigma \backslash \overline{\gamma}_c$. Thus, $(\tilde{u}^1, \tilde{u}^2, \tilde{v}, \tilde{w})$ is a element of the set K_{gc}.

Now let us show that $(\tilde{u}^1, \tilde{u}^2, \tilde{v}, \tilde{w})$ coincides with (u^1, u^2, v, w). Let $(\tilde{u}^1, \tilde{u}^2, \overline{v}, \overline{w}) \in K_{gc}$ be an arbitrary function. Substituting $(\tilde{u}^1 - u_p^1, \tilde{u}^2 - u_p^2, \overline{v} - v_p, \overline{w} - w_p)$ into (20) as a test function and taking into account (21)–(25), we obtain the variational inequality

$$\int\limits_{\Omega_1} \sigma(u_p^1) : \varepsilon(\tilde{u}^1 - u_p^1) dx + \int\limits_{\Omega_2} \sigma(u_p^2) : \varepsilon(\tilde{u}^2 - u_p^2) dx + \int\limits_\Sigma ESv_{p,1}(\overline{v}_{,1} - v_{p,1}) dx_1$$
$$+ \int\limits_\Sigma EIw_{p,11}(\overline{w}_{,11} - w_{p,11}) dx_1 \ge \int\limits_{\partial\Omega_1\cap\Gamma_N} f \cdot (\tilde{u}^1 - u_p^1) ds + \int\limits_{\partial\Omega_2\cap\Gamma_N} f \cdot (\tilde{u}^2 - u_p^2) ds. \quad (40)$$

In view of (38), passing to the limit as $p \to \infty$ in (40), we obtain a variational inequality of form (18) with (u^1, u^2, v, w) replaced by $(\tilde{u}^1, \tilde{u}^2, \tilde{v}, \tilde{w})$. Since inequality (18) has a unique solution, $(\tilde{u}^1, \tilde{u}^2, \tilde{v}, \tilde{w})$ coincides with (u^1, u^2, v, w). Moreover, it follows that (u_p^1, u_p^2, v_p, w_p) converges weakly to (u^1, u^2, v, w) in $U^1 \times U^2 \times V \times W$ as $p \to \infty$.

In addition to (38), we prove that, as $p \to \infty$,

$$(u_p^1, u_p^2, v_p, w_p) \to (u^1, u^2, v, w) \quad \text{strongly in} \quad U^1 \times U^2 \times V \times W. \tag{41}$$

From (18), we obtain

$$\int_{\Omega_1} \sigma(u^1) : \varepsilon(u^1) dx + \int_{\Omega_2} \sigma(u^2) : \varepsilon(u^2) dx + \int_{\Sigma} ESv_{,1} v_{,1} dx_1 + \int_{\Sigma} EIw_{,11} v_{,11} dx_1 = \int_{\partial\Omega_1 \cap \Gamma_N} f \cdot u^1 ds + \int_{\partial\Omega_2 \cap \Gamma_N} f \cdot u^2 ds. \tag{42}$$

Since the norm is weakly lower semicontinuous and in view of identity (42), the following chain of inequalities is valid:

$$\begin{aligned}\|u^1\|_{U^1}^2 + \|u^2\|_{U^2}^2 + \|v\|_V^2 + \|w\|_W^2 &\le \liminf_{p\to\infty} \left(\|u_p^1\|_{U^1}^2 + \|u_p^2\|_{U^2}^2 + \|v_p\|_V^2 + \|w_p\|_W^2 \right) \\ &\le \limsup_{p\to\infty} \left(\|u_p^1\|_{U^1}^2 + \|u_p^2\|_{U^2}^2 + \|v_p\|_V^2 + \|w_p\|_W^2 \right) = \|u^1\|_{U^1}^2 + \|u^2\|_{U^2}^2 + \|v\|_V^2 + \|w\|_W^2,\end{aligned}$$

whence, as $p \to \infty$,

$$\|u_p^1\|_{U^1}^2 + \|u_p^2\|_{U^2}^2 + \|v_p\|_V^2 + \|w_p\|_W^2 \to \|u^1\|_{U^1}^2 + \|u^2\|_{U^2}^2 + \|v\|_V^2 + \|w\|_W^2.$$

The weak convergence and convergence of the norms in $U^1 \times U^2 \times V \times W$ imply strong convergence (41). The theorem is completely proved.

6. ITERATIVE ALGORITHM

Let us construct an iterative algorithm for the numerical solution of problem (19). First, we note that variational equality (20) is equivalent to four ones:

$$\begin{aligned}&\int_{\Omega_1} \sigma(u_p^1) : \varepsilon(\bar{u}^1) dx + \int_{\gamma_c} \lambda^1 \bar{u}^1 \cdot \nu dx_1 + \int_{\Sigma} \lambda^2 \bar{u}^1 \cdot \tau dx_1 + \int_{\Sigma} \lambda^3 \bar{u}^1 \cdot \nu dx_1 \\ &+ \int_{\Sigma\setminus\bar{\gamma}_c} \lambda^4 \bar{u}^1 \cdot \nu dx_1 + \int_{\Sigma\setminus\bar{\gamma}_c} \lambda^5 \bar{u}^1 \cdot \tau dx_1 = \int_{\partial\Omega_1 \cap \Gamma_N} f \cdot \bar{u}^1 ds \quad \forall \bar{u}^1 \in U^1,\end{aligned}$$

$$\begin{aligned}&\int_{\Omega_2} \sigma(u_p^2) : \varepsilon(\bar{u}^2) dx - \int_{\gamma_c} \lambda^1 \bar{u}^2 \cdot \nu dx_1 - \int_{\Sigma\setminus\bar{\gamma}_c} \lambda^4 \bar{u}^2 \cdot \nu dx_1 - \int_{\Sigma\setminus\bar{\gamma}_c} \lambda^5 \bar{u}^2 \cdot \tau dx_1 \\ &= \int_{\partial\Omega_2 \cap \Gamma_N} f \cdot \bar{u}^2 ds \quad \forall \bar{u}^2 \in U^2,\end{aligned}$$

$$\int_{\Sigma} ESv_{p,1} \bar{v}_{,1} dx_1 - \int_{\Sigma} \lambda^2 \bar{v} dx_1 = 0 \quad \forall \bar{v} \in V,$$

$$\int_{\Sigma} EIw_{p,11} \bar{v}_{,11} dx_1 - \int_{\Sigma} \lambda^3 \bar{w} dx_1 = 0 \quad \forall \bar{w} \in W.$$

Let $P_{\Lambda_p^i}$ denote the projectors onto the sets Λ_p^i, $i = \overline{1,5}$, in the respective spaces $L_2(\gamma_c)$, $L_2(\Sigma)$, and $L_2(\Sigma\setminus\bar{\gamma}_c)$. The following Uzawa-type algorithm is proposed for solving problem (19).

Algorithm

Step 1. Iteration $k = 0$. Choose arbitrary $\lambda^{i,0} \in \Lambda_p^i$, $i = \overline{1,5}$. For example, it is possible to set $\lambda^{i,0} = 0$ for all $i = \overline{1,5}$.

Step 2. Iteration $k \geq 1$. Find the functions $u^{1,k}$, $u^{2,k}$, v^k, and w^k by solving the variational equalities

$$\int_{\Omega_1} \sigma(u^{1,k}) : \varepsilon(\bar{u}^1)dx + \int_{\gamma_c} \lambda^{1,k-1}\bar{u}^1 \cdot \nu dx_1 + \int_{\Sigma} \lambda^{2,k-1}\bar{u}^1 \cdot \tau dx_1 + \int_{\Sigma} \lambda^{3,k-1}\bar{u}^1 \cdot \nu dx_1$$

$$+ \int_{\Sigma\setminus\bar{\gamma}_c} \lambda^{4,k-1}\bar{u}^1 \cdot \nu dx_1 + \int_{\Sigma\setminus\bar{\gamma}_c} \lambda^{5,k-1}\bar{u}^1 \cdot \tau dx_1 = \int_{\partial\Omega_1\cap\Gamma_N} f \cdot \bar{u}^1 ds \quad \forall \bar{u}^1 \in U^1,$$

$$\int_{\Omega_2} \sigma(u^{2,k}) : \varepsilon(\bar{u}^2)dx - \int_{\gamma_c} \lambda^{1,k-1}\bar{u}^2 \cdot \nu dx_1 - \int_{\Sigma\setminus\bar{\gamma}_c} \lambda^{4,k-1}\bar{u}^2 \cdot \nu dx_1 - \int_{\Sigma\setminus\bar{\gamma}_c} \lambda^{5,k-1}\bar{u}^2 \cdot \tau dx_1$$

$$= \int_{\partial\Omega_2\cap\Gamma_N} f \cdot \bar{u}^2 ds \quad \forall \bar{u}^2 \in U^2,$$

$$\int_{\Sigma} ESv_{,1}^k \bar{v}_{,1} dx_1 - \int_{\Sigma} \lambda^{2,k-1}\bar{v} dx_1 = 0 \quad \forall \bar{v} \in V,$$

$$\int_{\Sigma} EIw_{,11}^k \bar{w}_{,11} dx_1 - \int_{\Sigma} \lambda^{2,k-1}\bar{w} dx_1 = 0 \quad \forall \bar{w} \in W.$$

Step 3. Terminate the algorithm or, for some $\theta > 0$, set

$$\lambda^{1,k+1} = P_{\Lambda_p^1}(\lambda^{1,k} + \theta(u^{1,k} - u^{2,k}) \cdot \nu),$$

$$\lambda^{2,k+1} = P_{\Lambda_p^2}(\lambda^{2,k} + \theta(u^{1,k} \cdot \tau - v^k)),$$

$$\lambda^{3,k+1} = P_{\Lambda_p^3}(\lambda^{3,k} + \theta(u^{1,k} \cdot \nu - w^k)),$$

$$\lambda^{4,k+1} = P_{\Lambda_p^4}(\lambda^{4,k} + \theta(u^{1,k} - u^{2,k}) \cdot \nu),$$

$$\lambda^{5,k+1} = P_{\Lambda_p^5}(\lambda^{5,k} + \theta(u^{1,k} - u^{2,k}) \cdot \tau).$$

The convergence of the iteration sequence follows from the general convergence theorems for the Uzawa algorithm (see [29, Chapter 7, Proposition 1.1]). Specifically, the following result holds.

Theorem 3. *There exists a number* $\theta^* > 0$ *such that, for all* $\theta \in (0, \theta^*)$, *as* $k \to \infty$,

$$(u^{1,k}, u^{2,k}, v^k, w^k) \to (u_p^1, u_p^2, v_p, w_p) \quad \textit{strongly in} \quad U^1 \times U^2 \times V \times W.$$

7. NUMERICAL EXPERIMENTS

Retaining the previous notation, we specify the computational domains Ω as the square $(-1,1)\times(-1,1)$ and define $\Sigma = (-1,1)\times\{0\}$ and $\gamma_c = (-1/2,1/2)\times\{0\}$. As before, homogeneous Dirichlet conditions are set on the right boundary of the square $\Gamma_D = \{1\}\times(-1,1)$, while Neumann conditions are set on the upper $\Gamma_N^1 = (-1,1)\times\{1\}$, lower $\Gamma_N^2 = (-1,1)\times\{-1\}$, and left $\Gamma_N^3 = \{-1\}\times(-1,1)$ boundaries.

In the examples considered below, we assume that the elastic part Ω_Σ is a homogeneous and isotropic one in a plane strain state. Then the stress and strain tensors are related by

$$\sigma_{11}(u) = (2\mu_m + \lambda_m)\varepsilon_{11}(u) + \lambda_m\varepsilon_{22}(u), \quad \sigma_{12}(u) = \sigma_{21}(u) = 2\mu_m\varepsilon_{12}(u),$$

$$\sigma_{22}(u) = \lambda_m\varepsilon_{11}(u) + (2\mu_m + \lambda_m)\varepsilon_{22}(u),$$

The elastic Lame constants μ_m and λ_m are expressed in terms of Young's modulus E_m and Poisson's ratio ν_m using the formulas

$$\mu_m = \frac{E_m}{2(1+\nu_m)}, \quad \lambda_m = \frac{2\nu_m\mu_m}{1-2\nu_m}.$$

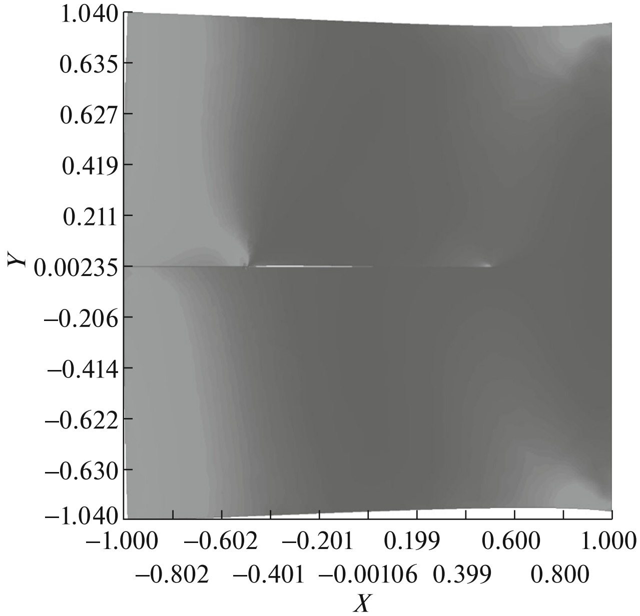

Fig. 2. Partial closure of the crack faces: deformed configuration and the von Mises stress distribution.

The parameter values are $\nu_m = 0.28$, $E_m = 200$ HPa, $ES = 390$ HPa m^2, and $EI = 39$ HPa m^4.

As was indicated above, the numerical computations were performed using the finite element method. The domain Ω_1 was partitioned into 13862 triangles with 7172 vertices, while Ω_2, into 13810 triangles with 7146 vertices. The interface Σ between the domains contained 400 vertices, and the minimum and maximum sizes of the triangles near Σ were 0.005 and 0.075, respectively. The stopping criterion for the algorithm was the condition

$$\max\left(\frac{\left\|u^{1,k}-u^{1,k-1}\right\|_{U^1}}{\left\|u^{1,k}\right\|_{U^1}}, \frac{\left\|u^{2,k}-u^{2,k-1}\right\|_{U^2}}{\left\|u^{2,k}\right\|_{U^2}}\right) < 10^{-7}.$$

In all numerical experiments, we used $p = 10^7$ and $\theta = 7$. This choice of the regularization parameter p was determined by estimate (36). To choose an optimal value of the relaxation parameter θ, the algorithm was tested on a coarse grid where the computation time was relatively insignificant and the resulting value of θ was then used on a fine grid. The spaces U^α, $\alpha = 1, 2$, were approximated by finite-element spaces consisting of piecewise smooth functions, i.e., Lagrange P_1-elements [30, 31].

7.1. Partial Closure of the Crack Faces

Setting $f|_{\Gamma_N^1} = 0.003\mu_m x$, $f|_{\Gamma_N^2} = -0.003\mu_m x$, and $f|_{\Gamma_N^3} = (0, 0)$, we see that the crack faces close in a neighborhood of the right tip of γ_c. Figure 2 shows the deformed configuration in the Lagrangian coordinates $x + 40u(x)$ with a scaling factor equal to 40 in both axes and the von Mises stress distribution (the second invariant of the stress tensor).

Figures 3 and 4 present the vertical and horizontal displacements of the body points lying on the crack faces Σ_1 and Σ_2 of the interface Σ between Ω_1 and Ω_2. Recall that by virtue of boundary conditions (10), the displacements of Σ_1 coincide with the vertical deflections and horizontal displacements of Σ. Note that the closure of the crack faces occurs on the segment [0.218, 0.5].

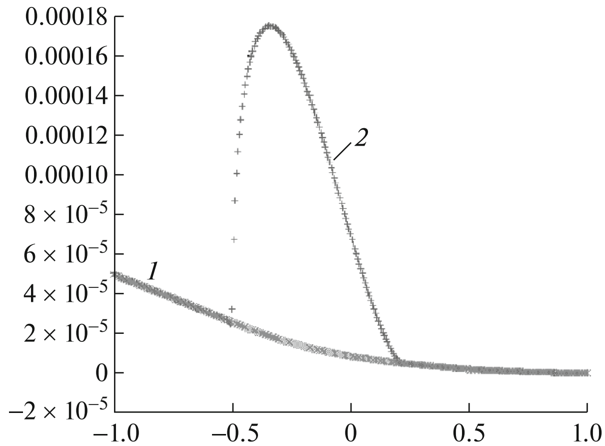

Fig. 3. Partial closure of the crack faces: vertical displacements of the interface.

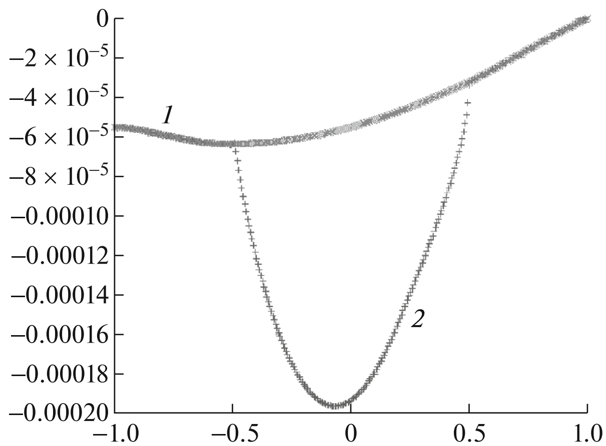

Fig. 4. Partial closure of the crack faces: horizontal displacements of the interface.

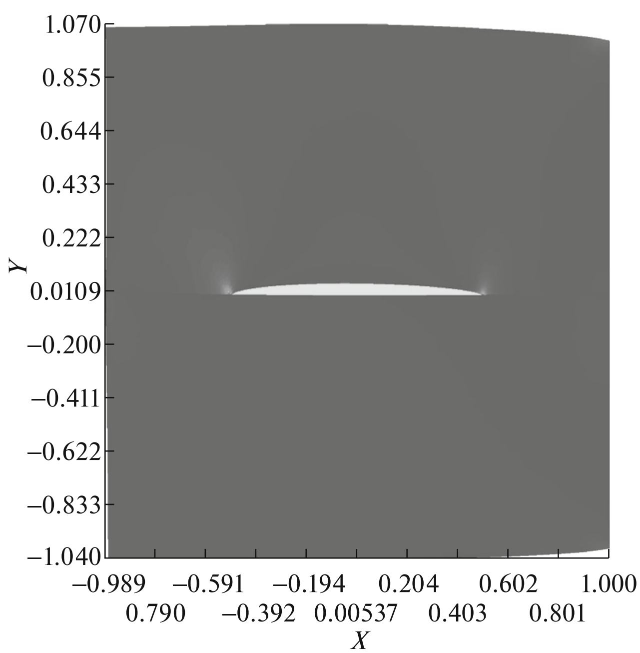

Fig. 5. Complete opening of the crack faces: the deformed configuration and the von Mises stress distribution.

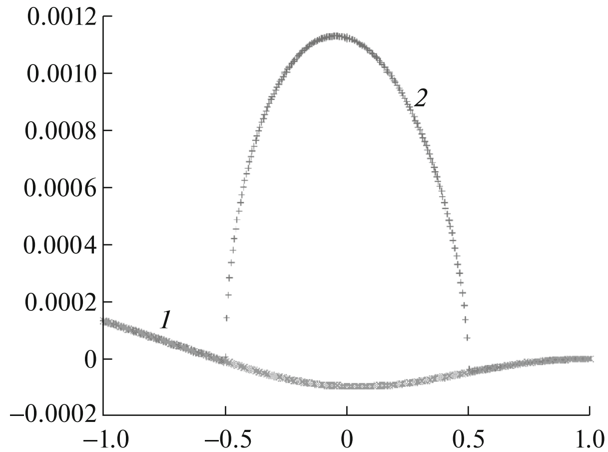

Fig. 6. Complete opening of the crack faces: vertical displacements of the interface.

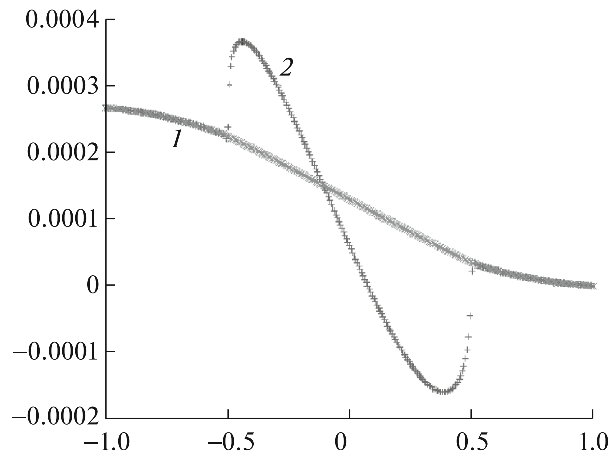

Fig. 7. Complete opening of the crack faces: horizontal displacements of the interface.

7.2. Complete Opening of the Crack Faces

Let $f|_{\Gamma_N^1} = (0, 0.003\mu_m)$, $f|_{\Gamma_N^2} = (0, 0.003\mu_m)$, and $f|_{\Gamma_N^3} = (0, 0)$. In this case, the faces of γ_c open completely. Figure 5 shows the deformed configuration in the Lagrangian coordinates $x + 40u(x)$ with a scaling factor equal to 40 in both axes and the von Mises stress distribution.

Figures 6 and 7 present the vertical and horizontal displacements of the body points lying on Σ_1 and Σ_2. As in the preceding example, the displacements of the points of Σ_1 are the vertical deflections and horizontal displacements of the points of Σ.

Analyzing the stress distributions near γ_c, we can conclude that the stress fields have a singularity at the crack tips. Moreover, a singularity occurs even when the crack faces are in contact near the crack tip. This finding is explained by the change in the boundary condition type at the crack tips. At the same time, it should be noted that the indicated stress singularity occurs only in Ω_2, but is absent in Ω_1. To the best of our knowledge, the exact asymptotics of the solution near the crack tips and, hence, the character of the stress singularity for model (1)–(11) remain an open question.

To conclude, we note that, in analyzing a stress singularity at a crack tip in fracture mechanics, it is useful to apply invariant energy integrals that are independent of a smooth curve enclosing the crack tip. For the model considered in this work, the existence of invariant integrals was established in [32].

ACKNOWLEDGMENTS

This work was supported by the Russian Science Foundation, project no. 15-11-10000.

REFERENCES

1. A. M. Khludnev and M. Negri, “Crack on the boundary of a thin elastic inclusion inside an elastic body,” Z. Angew. Math. Mech. **92** (5), 341–354 (2012).
2. A. M. Khludnev and G. Leugering, “Delaminated thin elastic inclusion inside elastic bodies,” Math. Mech. Complex Syst. **2** (1), 1–21 (2014).
3. H. Itou and A. M. Khludnev, “On delaminated thin Timoshenko inclusions inside elastic bodies,” Math. Methods Appl. Sci. **39** (17), 4980–4993 (2016).
4. A. M. Khludnev and G. Leugering, “On elastic bodies with thin rigid inclusions and cracks,” Math. Methods Appl. Sci. **33** (16), 1955–1967 (2010).
5. V. Shcherbakov, “Energy release rates for interfacial cracks in elastic bodies with thin semirigid inclusions,” Z. Angew. Math. Phys. **68**, Article 26 (2017).
6. E. Barbieri and N. M. Pugno, “A computational model for large deformations of composites with a 2D soft matrix and 1D anticracks,” Int. J. Solids Struct. **77**, 1–14 (2015).
7. L. G. S. Leite and W. S. Venturini, “Accurate modeling of rigid and soft inclusions in 2D elastic solids by the boundary element method,” Comput. Struct. **84** (29–30), 1874–1881 (2006).
8. E. M. Rudoy, “Numerical solution of an equilibrium problem for an elastic body with a thin delaminated rigid inclusion,” J. Appl. Ind. Math. **10** (2), 264–276 (2016).
9. E. M. Rudoy and V. V. Shcherbakov, “Domain decomposition method for a membrane with a delaminated thin rigid inclusion,” Sib. Electron. Math. Rep. **13**, 395–410 (2016).
10. E. M. Rudoy, “On numerical solving a rigid inclusions problem in 2D elasticity,” Z. Angew. Math. Phys. **68**, Article 19 (2017).
11. V. D. Korneev and V. M. Sveshnikov, “Parallel algorithms and domain decomposition techniques for solving three-dimensional boundary value problems on quasi-structured grids,” Numer. Anal. Appl. **9** (2), 141–149 (2016).
12. Yu. M. Laevskii and A. M. Matsokin, “Decomposition methods for the solution of elliptic and parabolic boundary value problems,” Sib. Zh. Vychisl. Mat. **2** (4), 361–372 (1999).
13. A. V. Rukavishnikov, “Domain decomposition method and numerical analysis of a fluid dynamics problem,” Comput. Math. Math. Phys. **54** (9), 1459–1480 (2014).
14. V. M. Sveshnikov, “Domain decomposition method in problems of high-current electronics,” Sib. Zh. Ind. Mat. **18** (2), 124–130 (2015).
15. T. K. Dobroserdova and M. A. Olshanskii, “A finite element solver and energy stable coupling for 3D and 1D fluid models,” Comput. Method. Appl. M **259**, 166–176 (2013).
16. J. Koko, “Uzawa conjugate gradient domain decomposition methods for coupled Stokes flows,” J. Sci. Comput. **26** (2), 195–215 (2006).
17. G. P. Astrakhantsev, “Domain decomposition method for the problem of bending heterogeneous plate,” Comput. Math. Math. Phys. **38** (10), 1686–1694 (1998).
18. G. Bayada, J. Sabil, and T. Sassi, “Convergence of a Neumann–Dirichlet algorithm for two-body contact problems with nonlocal Coulomb’s friction law,” ESAIM-Math. Model. Numer. **42** (2), 243–262 (2008).
19. J. Daněk, I. Hlaváček, and J. Nedoma, “Domain decomposition for generalized unilateral semi-coercive contact problem with given friction in elasticity,” Math. Comput. Simul. **68**, 271–300 (2005).
20. E. M. Rudoy, “Domain Decomposition method for crack problems with nonpenetration condition,” ESAIM-Math. Model. Numer. **50** (4), 995–1009 (2016).
21. E. M. Rudoy, N. A. Kazarinov, and V. Yu. Slesarenko, “Numerical simulation of equilibrium of an elastic two-layer structure with a through crack,” Numer. Anal. Appl. **10** (1), 63–73 (2017).
22. M. Hintermüller, V. Kovtunenko, and K. Kunisch, “The primal-dual active set method for a crack problem with non-penetration,” IMA J. Appl. Math. **69** (1), 1–26 (2004).
23. M. Hintermüller, K. Ito, and K. Kunisch, “The primal-dual active set strategy as a semismooth Newton method,” SIAM J. Optim. **13** (2), 865–888 (2003).
24. V. A. Kovtunenko, “Numerical simulation of the non-linear crack problem with nonpenetration,” Math. Methods Appl. Sci. **27** (2), 163–179 (2004).
25. S. R. Eugster, *Geometric Continuum Mechanics and Induced Beam Theories* (Springer, New York, 2015).
26. A. M. Khludnev and V. A. Kovtunenko, *Analysis of Cracks in Solids* (WIT, Southampton, 2000).
27. J. Céa, *Optimisation: Théorie Et Algorithmes* (Dunod, Paris, 1971).
28. E. M. Rudoy, “Domain decomposition method for a model crack problem with a possible contact of crack edges,” Comput. Math. Math. Phys. **55** (2), 305–306 (2015).
29. I. Ekeland and R. Temam, *Convex Analysis and Variational Problems* (SIAM, Philadelphia, 1999).
30. P. Ciarlet, *The Finite Element Method for Elliptic Problems* (North-Holland, Amsterdam, 1977).
31. G. Allaire, *Numerical Analysis and Optimization: An Introduction to Mathematical Modeling and Numerical Simulation* (Oxford University Press, Oxford, 2007).
32. A. M. Khludnev and V. V. Shcherbakov, “Singular path-independent energy integrals for elastic bodies with Euler–Bernoulli inclusions,” Math. Mech. Solids **22** (11), 2180–2195 (2017).

Translated by I. Ruzanova