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Numerical Simulation of Equilibrium of an Elastic Two-Layer Structure with a Through Crack

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Abstract—In this paper, a problem of equilibrium of two elastic bodies pasted together along a curve is considered. It is assumed that there is a through crack on a part of the curve. Nonlinear boundary conditions providing mutual non-penetration between the crack faces are set. The main objective of the paper is to construct and test a numerical algorithm for solving the equilibrium problem. The algorithm is based on two approaches: a domain decomposition method and Uzawa method for solving variational inequalities. A numerical experiment illustrates the efficiency of the algorithm.

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INTRODUCTION

Consider a two-layer structure of two elastic bodies pasted together along a line. In a part of the pasting line there is a through crack on which nonlinear boundary conditions of unilateral constraint are specified to exclude penetration of the crack faces into one another. The behavior of both layers of this structure is simulated as a plane problem in elasticity theory. The layers move identically along the pasting line and at the same crack faces. The structure is in equilibrium under the action of some forces applied to the external boundaries of the bodies.

The problem under consideration belongs to a class of crack theory problems with possible contact between the faces. Such problems are investigated, for instance, in [1–9]. Papers [10–12] are devoted to problems of equilibrium of multilayer structures with cracks. Specifically, in paper [12] correctness of the solution to the problem being considered in the present paper is proved and qualitative analysis of the solution is performed. Unfortunately, no results have been obtained so far for numerically solving the problems of equilibrium of multilayer structures with cracks and non-penetration. Some crack theory problems with non-penetration are solved numerically in [13–18].

The main purpose of the present paper is to create an efficient algorithm for solving a problem of equilibrium of the above two-layer structure with a through crack and non-penetration of the crack faces. Two approaches are used in constructing the algorithm: a domain decomposition method widely used for numerically solving various problems of fluid dynamics (see, for instance, [19–24]) and Uzawa method for solving problems with unilateral constraints. First, the initial domain in which the solution is sought for is divided into two subdomains. Then Lagrange multipliers are introduced to guarantee non-penetration of the crack faces and that the two layers move identically along the pasting line. Finally, examples of numerical calculations with finite element method are presented.

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1. PROBLEM STATEMENT

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_N \cup \overline{\Gamma}_D$, $\Gamma_N \cap \Gamma_D = \emptyset$ and $\operatorname{meas}\Gamma_D > 0$. Also, let $\gamma_c \subset \Omega$ be a smooth curve without self-crossings, $\overline{\gamma}_c \cap \overline{\Gamma}_D = \emptyset$. Assume that the curve Σ divides the domain Ω into two subdomains: Ω_- and Ω_+ with Lipschitz boundaries $\partial\Omega_-$ and $\partial\Omega_+$, respectively, and that the following conditions are satisfied:

$$\overline{\gamma}_c \subset \Sigma, \qquad \operatorname{meas}(\Sigma \setminus \overline{\gamma}_c) > 0, \qquad \operatorname{meas}(\partial\Omega_\pm \cap \Gamma_D) > 0.$$

Let us take a unit normal vector ν to Σ so that ν is the outward unit normal vector to Ω_-. Hence, the vector $(-\nu)$ is a unit normal vector to Ω_+ on Σ. Let τ denote the unit tangent vector on Σ. Also, let $\gamma_c^\pm$ denote the edge of the curve γ belonging to the subdomain boundary $\Omega_\pm$.

Let us denote $\Omega_\Sigma = \Omega \setminus \overline{\Sigma}$ and $\Omega_c = \Omega \setminus \overline{\gamma}_c$, and consider the following boundary value problem: For given vectors $f,\ g \in L_2(\Gamma_N)^2$, find vector-functions $u=(u_1, u_2)$ and $v = (v_1, v_2)$ such that

$$-\sigma_{ij,j}(u) = 0, \quad -\sigma_{ij,j}(v) = 0 \quad \text{in} \quad \Omega_\Sigma,\ i = 1,2, \tag{1}$$
$$u_i = 0,\ v_i = 0 \quad \text{on} \quad \Gamma_D, \tag{2}$$
$$\sigma_{ij}n_j = f_i, \quad p_{ij}n_j = g_i \quad \text{on} \quad \Gamma_N,\ i = 1,2, \tag{3}$$
$$u_i = v_i, \quad [\sigma_{ij}\nu_j + p_{ij}\nu_j] = 0 \quad \text{on} \quad \Sigma \setminus \overline{\gamma}_c,\ i = 1,2, \tag{4}$$
$$[u_j]\nu_j \ge 0 \quad \text{on} \quad \gamma_c, \tag{5}$$
$$u_i = v_i \quad \text{on} \quad \gamma_c^\pm,\ i = 1,2, \tag{6}$$
$$[\sigma_\nu(u) + p_\nu(v)] = 0, \quad (\sigma_\nu(u) + p_\nu(v))[u_j]\nu_j = 0 \quad \text{on} \quad \gamma_c, \tag{7}$$
$$\sigma_\tau(u) + p_\tau(v) = 0, \quad \sigma_\nu(u) + p_\nu(v) \le 0 \quad \text{on} \quad \gamma_c^\pm. \tag{8}$$

Here $u = (u_1, u_2)$, $v = (v_1, v_2)$ are displacements of the two elastic bodies (layers); $\sigma(u) = \{\sigma_{ij}(u)\}$ and $p(u) = \{p_{ij}(u)\}$ are the stress tensors of the elastic layers; $[w] = w|_{\gamma_c^+} - w|_{\gamma_c^-}$ is the jump of a function w on γ_c; n is the outward normal to $\partial\Omega$; $\sigma_\nu(u) = \sigma_{ij}(u)\nu_i\nu_j$; $\sigma_\tau(u) = \sigma_{ij}(u)\nu_i\tau_j$. The subscripts after comma denote the operations of differentiation with respect to the corresponding coordinates. Summation is implied over repeated indices. The quantities $p_\nu(v)$ and $p_\tau(v)$ are defined in the same way as $\sigma_\nu(u)$ and $\sigma_\tau(u)$.

Let $A = \{a_{ijkl}\}$, $B = \{b_{ijkl}\}$ $(i, j, k, l = 1, 2)$ be given tensors of elasticity coefficients satisfying the standard properties of symmetry and positive definiteness. Assume that linear Hooke's law is valid in the elastic layers:

$$\sigma_{ij}(u) = a_{ijkl}\varepsilon_{kl}(u), \quad p_{ij}(v) = b_{ijkl}\varepsilon_{kl}(v), \quad i, j = 1, 2,$$

where $\varepsilon(w) = \{\varepsilon_{ij}(w)\}$ is the linear strain tensor,

$$\varepsilon_{ij}(w) = \frac{1}{2}(w_{i,j} + w_{j,i}), \quad i, j = 1, 2.$$

The problem (1)–(8) describes a state of equilibrium of two elastic bodies pasted together along the curve Σ and rigidly fixed on Γ_D. On γ_c (part of the curve Σ) there is a through crack with non-penetration conditions on its faces.

2. VARIATIONAL FORMULATION OF THE PROBLEM

Let us formulate the problem (1)–(8) in variational form. For this, we define an energy functional

$$\Pi(u,v) = \frac{1}{2}\int\limits_{\Omega_c} \big(\sigma_{ij}(u)\varepsilon_{ij}(u) + p_{ij}(v)\varepsilon_{ij}(v)\big)\,dx - \int\limits_{\Gamma_N} \big(f_i u_i + g_i v_i\big)\,ds$$

and introduce a set of admissible displacements

$$K = \Big\{(u,v) \in H \mid [u_j]\,\nu_j \geq 0 \quad \text{almost everywhere on } \gamma_c, \quad u = v \text{ on } \gamma_c^{\pm}, \quad u = v \text{ on } \Sigma \setminus \overline{\gamma}_c\Big\},$$

where

$$H = H^1_{\Gamma_D}(\Omega)^2 \times H^1_{\Gamma_D}(\Omega)^2,$$

$$H^1_{\Gamma_D}(\Omega)^2 = \Big\{w \in H^1(\Omega_c)^2 \mid w_1 = w_2 = 0 \quad \text{almost everywhere on } \Gamma_D\Big\}.$$

The boundary value problem (1)–(8) is formulated as a minimization problem: Find a pair of functions $(u,v) \in K$ minimizing the energy functional Π on the set K:

$$\Pi(u,v) = \inf_{\bar{u},\bar{v} \in K} \Pi(\bar{u},\bar{v}). \tag{9}$$

It is well known (see, for instance, [12]) that the problem (9) has a unique solution satisfying the variational inequality

$$\int\limits_{\Omega_c} \big(\sigma_{ij}(u)\varepsilon_{ij}(\bar{u}-u) + p_{ij}(v)\varepsilon_{ij}(\bar{v}-v)\big)\,dx \geq \int\limits_{\Gamma_N} \big(f_i(\bar{u}_i - u_i) + g_i(\bar{v}_i - v_i)\big)\,ds \quad \forall\,(\bar{u},\bar{v}) \in K.$$

It is also shown in [12] that the differential formulation (1)–(8) and the variational one (9) of the equilibrium problem are equivalent.

3. DOMAIN DECOMPOSITION

Let us define functional spaces:

$$U^{\pm} = \big\{u^{\pm} \in H^1(\Omega_{\pm})^2 \mid u_1^{\pm} = u_2^{\pm} = 0 \quad \text{almost everywhere on } \partial\Omega_{\pm} \cap \Gamma_D\big\},$$

$$V^{\pm} = \big\{v^{\pm} \in H^1(\Omega_{\pm})^2 \mid v_1^{\pm} = v_2^{\pm} = 0 \quad \text{almost everywhere on } \partial\Omega_{\pm} \cap \Gamma_D\big\}$$

and a set $K_{gc} \subset U^- \times U^+ \times V^- \times V^+$:

$$K_{gc} = \Big\{(u^-,u^+,v^-,v^+) \in U^- \times U^+ \times V^- \times V^+ \mid (u^+ - u^-)\nu \geq 0 \quad \text{almost everywhere on } \gamma_c,$$

$$u^- - u^+ = 0 \quad \text{on } \Sigma \setminus \overline{\gamma}_c, \quad u^- - v^- = 0,\ u^+ - v^+ = 0 \quad \text{almost everywhere on } \Sigma\Big\}.$$

Since we assume that meas $(\partial\Omega_{\pm} \cap \Gamma_D) > 0$, by virtue of the Korn inequality a norm in the spaces $U^{\pm}$, $V^{\pm}$ can be defined by the formula

$$\|w^{\pm}\|^2 = \int\limits_{\Omega_{\pm}} \sigma_{ij}(w^{\pm})\varepsilon_{ij}(w^{\pm})\,dx,$$

where $w^{\pm}$ belongs to the spaces $U^{\pm}$ or $V^{\pm}$.

On $U^{\pm}$ and $V^{\pm}$, we define the corresponding energy functionals $\Pi_{\pm}(u^{\pm})$ and $\Pi_{\pm}(v^{\pm})$ and consider the following minimization problem: Find functions $(u^-, u^+, v^-, v^+) \in K_{\mathrm{gc}}$ such that

$$\Pi_-(u^-)+\Pi_+(u^+)+\Pi_-(v^-)+\Pi_+(v^+)$$
$$= \inf_{(\bar{u}^-,\bar{u}^+,\bar{v}^-,\bar{v}^+)\in K_{\mathrm{gc}}} \Big(\Pi_-(u^-)+\Pi_+(u^+)+\Pi_-(v^-)+\Pi_+(v^+)\Big). \tag{10}$$

The following theorem holds (see, for instance, [13, 14]):

Theorem 1. *The problem* (10) *has a unique solution* $(u^-, u^+, v^-, v^+) \in K_{\mathrm{gc}}$. *In addition,*

$$u^{\pm} = u\,|_{\Omega_{\pm}}, \quad v^{\pm} = v\,|_{\Omega_{\pm}},$$

where (u, v) *is the solution to the problem* (9).

4. REDUCING THE PROBLEM TO FINDING A SADDLE POINT

For an arbitrary number $p > 0$, we define the following sets:

$$\begin{aligned}
U_p^{\pm} &= \Big\{u^{\pm} \in U^{\pm} \mid \|u^{\pm}\|_{U^{\pm}} \le p\Big\},\\
V_p^{\pm} &= \Big\{v^{\pm} \in V^{\pm} \mid \|v^{\pm}\|_{V^{\pm}} \le p\Big\},\\
\Lambda_p^{\mathrm{c}} &= \Big\{\lambda^{\mathrm{c}} \in L_2(\gamma_{\mathrm{c}}) \mid 0 \le \lambda^{\mathrm{c}} \le p \quad \text{almost everywhere on } \gamma_{\mathrm{c}}\Big\},\\
\Lambda_p^{\mathrm{g}} &= \Big\{\lambda^{\mathrm{g}} \in L_2(\Sigma \setminus \overline{\gamma}_{\mathrm{c}})^2 \mid -p \le \lambda_1^{\mathrm{g}},\ \lambda_2^{\mathrm{g}} \le p \quad \text{almost everywhere on } \Sigma \setminus \overline{\gamma}_{\mathrm{c}}\Big\},\\
\Lambda_p^{-} &= \Big\{\lambda^{-} \in L_2(\Sigma)^2 \mid -p \le \lambda_1^{-},\ \lambda_2^{-} \le p \quad \text{almost everywhere on } \Sigma\Big\},\\
\Lambda_p^{+} &= \Big\{\lambda^{+} \in L_2(\Sigma)^2 \mid -p \le \lambda_1^{+},\ \lambda_2^{+} \le p \quad \text{almost everywhere on } \Sigma\Big\}.
\end{aligned}$$

On the set $U_p^- \times U_p^+ \times V_p^- \times V_p^+ \times \Lambda_p^{\mathrm{c}} \times \Lambda_p^{\mathrm{g}} \times \Lambda_p^- \times \Lambda_p^+$, we define a Lagrange function

$$L(u^-, u^+, v^-, v^+, \lambda^{\mathrm{c}}, \lambda^{\mathrm{g}}, \lambda^-, \lambda^+)$$
$$= \Pi_-(u^-)+\Pi_+(u^+)+\Pi_-(v^-)+\Pi_+(v^+) + \int\limits_{\gamma_{\mathrm{c}}} \lambda^{\mathrm{c}}(u_i^- - u_i^+)\nu_i\,ds$$
$$+ \int\limits_{\Sigma\setminus\overline{\gamma}_{\mathrm{c}}} \lambda_i^{\mathrm{g}}(u_i^- - u_i^+)\,ds + \int\limits_{\Sigma} \lambda_i^-(u_i^- - v_i^-)\,ds + \int\limits_{\Sigma} \lambda_i^+(u_i^+ - v_i^+)\,ds,$$

associated with a family of problems of finding a saddle point of the Lagrangian L: Find functions $(u_p^-, u_p^+, v_p^-, v_p^+, \mu_p^{\mathrm{c}}, \mu_p^{\mathrm{g}}, \mu_p^-, \mu_p^+) \in U_p^- \times U_p^+ \times V_p^- \times V_p^+ \times \Lambda_p^{\mathrm{c}} \times \Lambda_p^{\mathrm{g}} \times \Lambda_p^- \times \Lambda_p^+$ such that

$$L\big(u_p^-, u_p^+, v_p^-, v_p^+, \bar\mu^{\,\mathrm{c}}, \bar\mu^{\,\mathrm{g}}, \bar\mu^{\,-}, \bar\mu^{\,+}\big) \le L\big(u_p^-, u_p^+, v_p^-, v_p^+, \mu_p^{\mathrm{c}}, \mu_p^{\mathrm{g}}, \mu_p^-, \mu_p^+\big)$$

$$\le L\big(\bar u^{\,-}, \bar u^{\,+}, \bar v^{\,-}, \bar v^{\,+}, \mu_p^{\mathrm{c}}, \mu_p^{\mathrm{g}}, \mu_p^-, \mu_p^+\big) \tag{11}$$

for all $\big(\bar u^{\,-}, \bar u^{\,+}, \bar v^{\,-}, \bar v^{\,+}, \bar\mu^{\,\mathrm{c}}, \bar\mu^{\,\mathrm{g}}, \bar\mu^{\,-}, \bar\mu^{\,+}\big) \in U_p^- \times U_p^+ \times V_p^- \times V_p^+ \times \Lambda_p^{\mathrm{c}} \times \Lambda_p^{\mathrm{g}} \times \Lambda_p^- \times \Lambda_p^+$.

Note that the Lagrange multiplier λ^{c} guarantees non-penetration on the crack γ_{c}. The multipliers $\lambda^\pm$ guarantee that the two bodies are pasted together along the curve Σ, and λ^{g} guarantees that the two subdomains, Ω_- and Ω_+, into which the domain Ω is divided are pasted together.

Since the sets $U_p^\pm$, $V_p^\pm$, Λ_p^{c}, Λ_p^{g}, and $\Lambda_p^\pm$ are convex closed ones that are bounded in the corresponding Banach spaces, the Lagrangian L is convex and lower semi-continuous with respect to (u^-, u^+, v^-, v^+) and concave and upper semi-continuous with respect to $(\lambda^{\mathrm{c}}, \lambda^{\mathrm{g}}, \lambda^-, \lambda^+)$, the problem (11) has a solution for all $p > 0$. The following theorems hold:

Theorem 2. *There exists a constant c such that for all $p > c$ the saddle point $\big(u_p^-, u_p^+, v_p^-, v_p^+, \mu_p^{\mathrm{c}}, \mu_p^{\mathrm{g}}, \mu_p^\nu, \mu_p^\tau\big)$ satisfies the following system of variational equalities and inequalities*:

$$\int\limits_{\Omega_-} \sigma_{ij}(u_p^-)\varepsilon_{ij}(\bar u^{\,-})\,dx + \int\limits_{\gamma_{\mathrm{c}}} \mu_p^{\mathrm{c}} \bar u_i^{\,-} \nu_i\,ds + \int\limits_{\Sigma\setminus\overline{\gamma}_{\mathrm{c}}} \mu_{pi}^{\mathrm{g}} \bar u_i^{\,-}\,ds + \int\limits_{\Sigma} \mu_{pi}^- \bar u_i^{\,-}\,ds = \int\limits_{\Gamma_N\cap\partial\Omega_-} f_i \bar u_i^{\,-}\,ds \quad \forall\,\bar u^{\,-} \in U^-,$$

$$\int\limits_{\Omega_+} \sigma_{ij}(u_p^+)\varepsilon_{ij}(\bar u^{\,+})\,dx - \int\limits_{\gamma_{\mathrm{c}}} \mu_p^{\mathrm{c}} \bar u_i^{\,+} \nu_i\,ds - \int\limits_{\Sigma\setminus\overline{\gamma}_{\mathrm{c}}} \mu_{pi}^{\mathrm{g}} \bar u_i^{\,+}\,ds + \int\limits_{\Sigma} \mu_{pi}^+ \bar u_i^{\,+}\,ds = \int\limits_{\Gamma_N\cap\partial\Omega_+} f_i \bar u_i^{\,+}\,ds \quad \forall\,\bar u^{\,+} \in U^+,$$

$$\int\limits_{\Omega_-} p_{ij}(v_p^-)\varepsilon_{ij}(\bar v^{\,-})\,dx - \int\limits_{\Sigma} \mu_{pi}^- \bar v^{\,-}\,ds = \int\limits_{\Gamma_N\cap\partial\Omega_-} g_i \bar v_i^{\,-}\,ds \quad \forall\,\bar v^{\,-} \in V^-,$$

$$\int\limits_{\Omega_+} p_{ij}(v_p^+)\varepsilon_{ij}(\bar v^{\,+})\,dx - \int\limits_{\Sigma} \mu_{pi}^+ \bar v^{\,+}\,ds = \int\limits_{\Gamma_N\cap\partial\Omega_+} g_i \bar v_i^{\,+}\,ds \quad \forall\,\bar v^{\,+} \in V^+,$$

$$\int\limits_{\gamma_{\mathrm{c}}} \lambda^{\mathrm{c}}(u_{pi}^- - u_{pi}^+)\nu_i\,ds \le \int\limits_{\gamma_{\mathrm{c}}} \mu_p^{\mathrm{c}}(u_{pi}^- - u_{pi}^+)\nu_i\,ds \quad \forall\,\lambda^{\mathrm{c}} \in \Lambda_p^{\mathrm{c}},$$

$$\int\limits_{\Sigma\setminus\overline{\gamma}_{\mathrm{c}}} \lambda_i^{\mathrm{g}}(u_{pi}^- - u_{pi}^+)\,ds \le \int\limits_{\Sigma\setminus\overline{\gamma}_{\mathrm{c}}} \mu_{pi}^{\mathrm{g}}(u_{pi}^- - u_{pi}^+)\,ds \quad \forall\,\lambda^{\mathrm{g}} \in \Lambda_p^-,$$

$$\int\limits_{\Sigma} \lambda_i^-(u_{pi}^- - v_{pi}^-)\,ds \le \int\limits_{\Sigma} \mu_{pi}^-(u_{pi}^- - v_{pi}^-)\,ds \quad \forall\,\lambda^- \in \Lambda_p^-,$$

$$\int\limits_{\Sigma} \lambda_i^+(u_{pi}^+ - v_{pi}^+)\,ds \le \int\limits_{\Sigma} \mu_{pi}^+(u_{pi}^+ - v_{pi}^+)\,ds \quad \forall\,\lambda^+ \in \Lambda_p^+.$$

Theorem 3. *As $p \to \infty$, we have convergence*

$$\big(u_p^-, u_p^+, v_p^-, v_p^+\big) \to \big(u^-, u^+, v^-, v^+\big) \quad \textit{that is strong in} \quad U^- \times U^+ \times V^- \times V^+.$$

Theorems 2 and 3 are proved similarly to those in [13, 14, 24]. In these papers, a domain decomposition method for various models of elastic bodies with cracks with possible contact between the faces is proposed.

The following estimate follows from [13, 14]:

$$0 \le \int\limits_{\gamma_c} h_+\big((u_{pi}^- - u_{pi}^+)\nu_i\big)\,ds + \int\limits_{\Sigma\setminus\overline{\gamma}_c} \big(h_+(u_{pi}^- - u_{pi}^+) + h_-(u_{pi}^- - u_{pi}^+)\big)\,ds$$

$$+ \int\limits_{\Sigma}\big(h_+(u_{pi}^- - v_{pi}^-) + h_-(u_{pi}^- - v_{pi}^-)\big)\,ds + \int\limits_{\Sigma}\big(h_+(u_{pi}^+ - v_{pi}^+) + h_-(u_{pi}^+ - v_{pi}^+)\big)\,ds \le \frac{K}{p}\,, \qquad (12)$$

where K is a constant depending on the functions f and g; h_+ and h_- have the form

$$h_+(w)(x) = \begin{cases} w(x) & \text{if } \ v(x) \ge 0, \\ 0 & \text{if } \ w(x) < 0, \end{cases} \qquad h_-(w)(x) = h_+(w)(x) - w(x).$$

Note that if, for instance, $h_+\big(u_{pi}^- - u_{pi}^+\big) > 0$ at some point $x \in \Sigma \setminus \overline{\gamma}_c$, the subdomains Ω_- and Ω_+ "diverge" at this point, and the condition $h_-\big(u_{pi}^+ - v_{pi}^+\big) > 0$ indicates that the subdomains Ω_- and Ω_+ overlap. Thus, the estimate (12) makes it possible to assess the accuracy of approximation of the solution to the problem (10) by the solutions to the problems of finding saddle points (11).

5. ITERATIVE ALGORITHM FOR SOLVING THE PROBLEM

Let $p > c$, where c is the constant from Theorem 2. Let $P_{\Lambda_p^c}$ and $P_{\Lambda_p^\pm}$ denote the operators of projection onto the sets Λ_p^c, Λ_p^g, and $\Lambda_p^\pm$, respectively, which have the following simple form in the spaces $L_2(\gamma_c)$, $L_2(\Sigma\setminus\overline{\gamma}_c)^2$, and $L_2(\Sigma)^2$:

$$P_{\Lambda_p^c} w(x) = \begin{cases} 0 & \text{if } \ w(x) \le 0, \\ w(x) & \text{if } \ 0 < w(x) < p\,, \\ p & \text{if } \ w(x) \ge p\,, \end{cases}$$

$$P_{\Lambda_p^g}(w_1, w_2) = P_{\Lambda_p^\pm}(w_1, w_2) = \Big(P(w_1), P(w_2)\Big),$$

where

$$P(w_i)(x) = \begin{cases} -p & \text{if } \ w_i(x) \le -p\,, \\ w_i(x) & \text{if } \ -p < w_i(x) < p\,, \\ p & \text{if } \ w_i(x) \ge p\,, \end{cases} \qquad i = 1, 2.$$

To solve the problem (11), which approximates (1)–(8), we propose the following Uzawa algorithm:

1. Iteration $k = 0$. Specify arbitrarily $\mu^{c,0} \in \Lambda_p^c$, $\mu^{g,0} \in \Lambda_p^g$, $\mu^{-,0} \in \Lambda_p^-$, and $\mu^{+,0} \in \Lambda_p^+$.

2. For every $k \geq 0$, find $(u^{-,k}, u^{+,k}, v^{-,k}, v^{+,k})$ as solutions to the following linear problems:

$$\int_{\Omega_-} \sigma_{ij}(u^{-,k})\varepsilon_{ij}(\bar{u}^-)\,dx + \int_{\gamma_c} \mu^{c,k}\bar{u}_i^-\nu_i\,ds + \int_{\Sigma\setminus\overline{\gamma}_c} \mu_{pi}^{g,k}\bar{u}_i^-\,ds + \int_{\Sigma} \mu_i^{-,k}\bar{u}_i^-\,ds$$

$$= \int_{\Gamma_N\cap\partial\Omega_-} f_i\bar{u}_i^-\,ds \quad \forall\,\bar{u}^- \in U^-,$$

$$\int_{\Omega_+} \sigma_{ij}(u^{+,k})\varepsilon_{ij}(\bar{u}^+)\,dx - \int_{\gamma_c} \mu^{c,k}\bar{u}_i^+\nu_i\,ds - \int_{\Sigma\setminus\overline{\gamma}_c} \mu_{pi}^{g,k}\bar{u}_i^+\,ds + \int_{\Sigma} \mu_i^{+,k}\bar{u}_i^+\,ds$$

$$= \int_{\Gamma_N\cap\partial\Omega_+} f_i\bar{u}_i^+\,ds \quad \forall\,\bar{u}^+ \in U^+,$$

$$\int_{\Omega_-} p_{ij}(v^{-,k})\varepsilon_{ij}(\bar{v}^-)\,dx - \int_{\Sigma} \mu_i^{-,k}\bar{v}_i^-\,ds = \int_{\Gamma_N\cap\partial\Omega_-} g_i\bar{v}_i^-\,ds \quad \forall\,\bar{v}^- \in V^-,$$

$$\int_{\Omega_+} p_{ij}(v^{+,k})\varepsilon_{ij}(\bar{v}^+)\,dx - \int_{\Sigma} \mu_i^{+,k}\bar{v}_i^+\,ds = \int_{\Gamma_N\cap\partial\Omega_+} g_i\bar{v}_i^+\,ds \quad \forall\,\bar{v}^+ \in V^+.$$

3. Define $\mu^{c,k+1}$, $\mu^{-,k+1}$, and $\mu^{+,k+1}$ by the formulas

$$\mu^{c,k+1} = P_{\Lambda_p^c}\Big(\mu^{c,k} + \theta\big(u_i^{-,k} - u_i^{+,k}\big)\nu_i\Big),$$
$$\mu^{g,k+1} = P_{\Lambda_p^g}\Big(\mu^{g,k} + \theta\big(u^{-,k} - u^{+,k}\big)\Big),$$
$$\mu^{-,k+1} = P_{\Lambda_p^-}\Big(\mu^{-,k} + \theta\big(u^{-,k} - v^{-,k}\big)\Big),$$
$$\mu^{+,k+1} = P_{\Lambda_p^+}\Big(\mu^{+,k} + \theta\big(u^{+,k} - v^{+,k}\big)\Big).$$

4. Stop, otherwise $k = k+1$, and go to step 2.

The convergence of the sequence $\big(u^{-,k}, u^{+,k}, v^{-,k}, v^{+,k}\big)$ to the solution $\big(u_p^-, u_p^+, v_p^-, v_p^+\big)$ of the regularized problem (11) as $k \to \infty$ follows from general theorems of convergence of Uzawa-type algorithms (see, for instance, [25, 26]). Therefore, the following theorem is valid.

Theorem 4. *There exists a number $\theta^* > 0$ such that for all $\theta \in (0, \theta^*)$ the sequence $\big(u^{-,k}, u^{+,k}, v^{-,k}, v^{+,k}\big)$ strongly converges to $\big(u_p^-, u_p^+, v_p^-, v_p^+\big)$ in $U^- \times U^+ \times V^- \times V^+$ as $k \to \infty$.*

Remark. The number θ^* depends on the norms of trace operators acting from $H^1_{\Gamma_D}(\Omega_\pm)$ to $L_2(\Sigma)$.

6. EXAMPLES OF NUMERICAL CALCULATIONS

Let $\Omega = (-1,1) \times (-1,1)$ be a square domain divided into two subdomains as follows:

$$\Omega_- = (-1,1) \times (-1,0), \qquad \Omega_+ = (-1,1) \times (0,1)$$

with a common boundary

$$\Sigma = (-1,1) \times \{0\}.$$

Assume that $\gamma_c = (-1/2, 1/2) \times \{0\}$ is a crack lying on Σ. Also assume that on

$$\Gamma_D = \big(\{-1\} \cup \{1\}\big) \times (-1,1)$$

both bodies are fixed. Let $\Gamma_N^+ = (-1,1) \times \{1\}$ and $\Gamma_N^- = (-1,1) \times \{-1\}$ denote the upper and lower boundaries of the square Ω, respectively.

Let both bodies be isotropic and homogeneous, that is, the following relations are valid:

$$\begin{aligned}
&\sigma_{11}(u) = (2\mu_1 + \lambda_1)\varepsilon_{11}(u) + \lambda_1\varepsilon_{22}(u), && p_{11}(v) = (2\mu_2 + \lambda_2)\varepsilon_{11}(v) + \lambda_2\varepsilon_{22}(v),\\
&\sigma_{12}(u) = \sigma_{21}(u) = 2\mu_1\varepsilon_{12}(u), && p_{12}(v) = p_{21}(v) = 2\mu_2\varepsilon_{12}(v),\\
&\sigma_{22}(u) = \lambda_1\varepsilon_{11}(u) + (2\mu_1 + \lambda_1)\varepsilon_{22}(u), && p_{22}(v) = \lambda_2\varepsilon_{11}(v) + (2\mu_2 + \lambda_2)\varepsilon_{22}(v),
\end{aligned}$$

where

$$\mu_i = \frac{E_i}{2(1+\nu_i)}\,, \qquad \lambda_i = \frac{2\nu_i\mu_i}{1-2\nu_i}\,, \quad i = 1,2.$$

We take the following material parameter values:

$$\begin{aligned}
&\nu_1 = 0.28, && E_1 = 200\,\Gamma\mathrm{GPa},\\
&\nu_2 = 0.32, && E_2 = 112\,\Gamma\mathrm{GPa}.
\end{aligned}$$

It is assumed that in all the numerical experiments $p = 10^7$. Also, as a termination criterion of the algorithm, we take

$$\max\left(\frac{\|u^{-,k} - u^{-,k-1}\|_{U^-}}{\|u^{-,k}\|^2_{U^-}},\ \frac{\|u^{+,k} - u^{+,k-1}\|_{U^+}}{\|u^{+,k}\|^2_{U^+}},\ \frac{\|v^{-,k} - v^{-,k-1}\|_{V^-}}{\|v^{-,k}\|^2_{V^-}},\ \frac{\|v^{+,k} - v^{+,k-1}\|_{V^+}}{\|v^{+,k}\|^2_{V^+}}\right) < 10^{-6}.$$

The spaces $U^\pm$ and $V^\pm$ are approximated by finite element spaces consisting of piecewise linear functions, $P1$-Lagrange elements (see [27, 28]).

Example 1. *Partial crack closure*. Assume that external loads $f = 10^{-3}\mu_1 x$ on Γ_N^- and $f = -10^{-3}\mu_1 x$ on Γ_N^+ are applied to one body (material parameters with index 1), whereas the other body is without load, that is, $g = 0$ on $\Gamma_N^- \cup \Gamma_N^+$.

Let P denote the number of triangle vertices lying on Σ, and let M be the number of triangle vertices on the external boundary $\partial\Omega$. Let $\theta = 65$. The table presents the results of executing the algorithm given in Section 5 for various triangulations of the subdomains Ω_- and Ω_+, $h^\pm_{\min}$ and $h^\pm_{\max}$ denote the minimal

Calculation results

P	M	$h^-_{\min}$	$h^-_{\max}$	$h^+_{\min}$	$h^+_{\max}$	Nod^-	$Triang^-$	Nod^+	$Triang^+$	$iter$
12	32	0.172	0.358	0.202	0.358	68	106	67	104	843
24	48	0.081	0.260	0.085	0.239	163	276	164	268	1164
48	80	0.041	0.155	0.041	0.151	498	906	498	906	1416
96	144	0.018	0.084	0.021	0.086	1733	3296	1716	3262	1909
200	256	0.0091	0.045	0.0098	0.047	6056	11782	6133	11936	2402

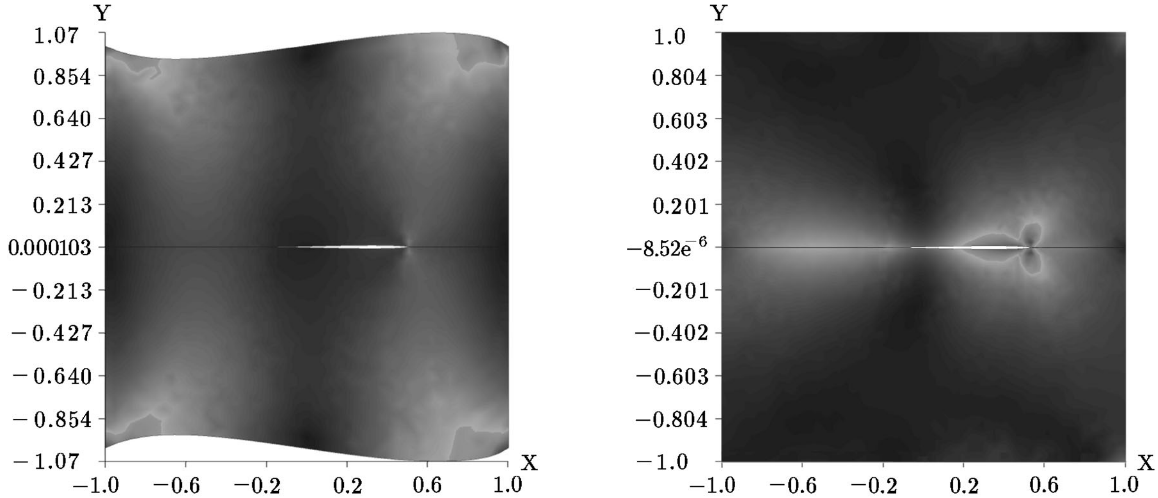

Fig. 1. Partial crack closure.

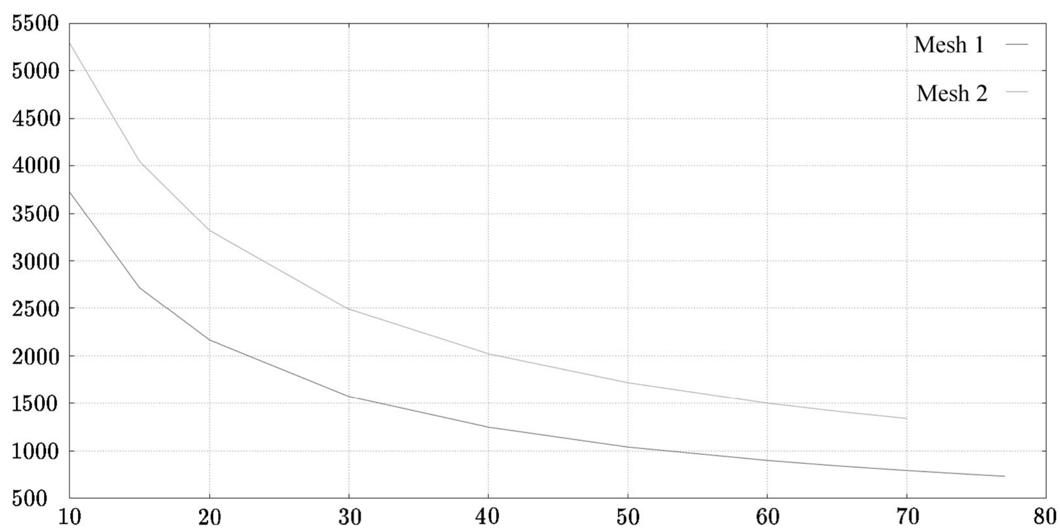

Fig. 2. The number of iterations versus the parameter θ.

and maximal sizes of triangulation cells, $Nod^{\pm}$, the number of triangle vertices, $Triang^{\pm}$, the number of triangles in the domains $\Omega_{\pm}$, and $iter$, the number of iterations to terminate the algorithm.

Figure 1 shows the deformed bodies with an increasing coefficient of 500 in both axes and distributions of von Mises stresses (left: body with load, right: body without load).

Figure 2 presents the number of iterations versus the relaxation parameter θ for two partitions of the domain Ω: $P/M = 12/32$ (curve Mesh 1) and $P/M = 48/80$ (curve Mesh 2). The curves show that as θ increases, the number of iterations decreases. For Mesh 1, the algorithm starts to diverge at $\theta = 78$, and for Mesh 2, it diverges already at $\theta = 75$. Note that at $\theta < 10$ the algorithm converges, but the number of iterations increases considerably.

Thus, comparing the curves, we can give the following recommendations for choosing an optimal value of the relaxation parameter θ: First, the algorithm should be tested on a coarse mesh for which the running time is small, and then the thus-obtained values of the parameter θ should be used on a fine mesh.

It was mentioned in the introduction that in the case of linear boundary conditions on crack faces there exist solutions at which the crack faces penetrate into each other. Figure 3 shows a deformed body

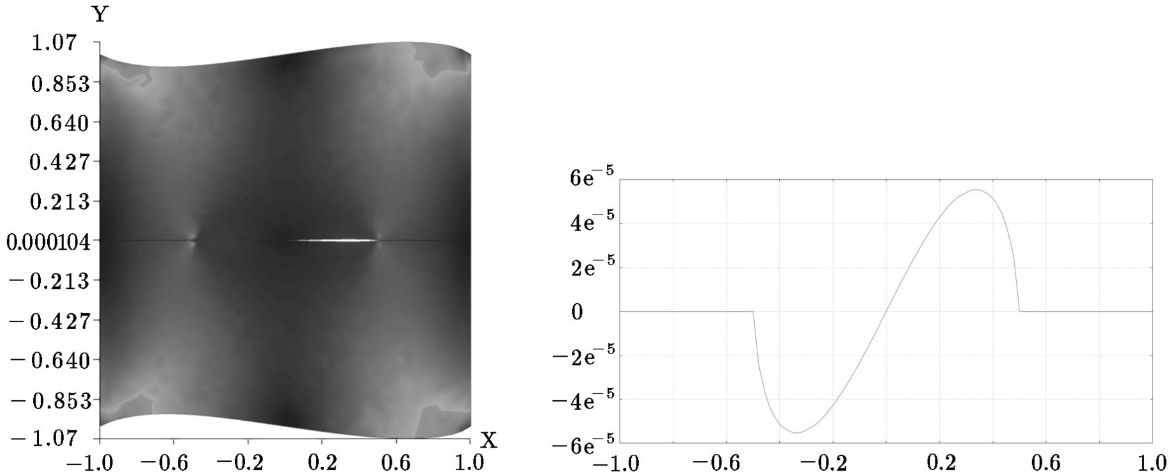

Fig. 3. Linear problem.

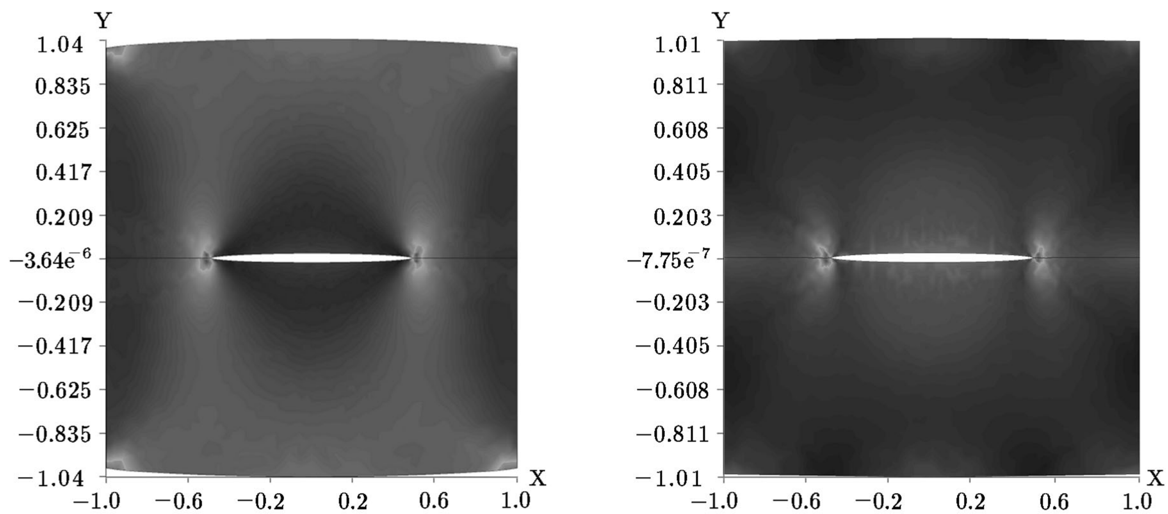

Fig. 4. Full crack opening.

(left: a domain after deformation and the distributions of von Mises stresses, right: the jump along the pasting line of the bodies Σ) with the same load but without the condition of non-penetration between the crack faces. The domain triangulation corresponds to $P = 96$ and $M = 144$, and the number of iterations $iter = 1931$.

Example 2. *Full crack opening.* Assume that the following forces are applied to $\Gamma_N^{\pm}$: for the first body, $f = -10^{-3}\mu_1$ on Γ_N^- and $f = 10^{-3}\mu_1$ on Γ_N^+; the other body is without load, that is, $g = 0$ on $\Gamma_N^- \cup \Gamma_N^+$.

The configurations of the bodies after deformations (with an increasing coefficient of 100 in both axes) and the distributions of von Mises stresses are shown in Fig. 4 (left: first body, right: second body). In this example $P = 96$, $M = 144$, and the number of iterations $iter = 2113$.

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